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HARVARD UNIVERSITY

DEPARTMENT
OF
MATHEMATICS

NEWCOMB'S MATHEMATICAL COURSE

ELEMENTS
OF
PLANE AND SPHERICAL TRIGONOMETRY

WITH

LOGARITHMIC AND OTHER MATHEMATICAL TABLES

AND EXAMPLES OF THEIR USE AND HINTS ON THE ART OF COMPUTATION

BY

SIMON NEWCOMB
Professor of Mathematics, U.S.N.

NEW YORK
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PREFACE.

The distinctive features of the following work belong partly to the course of which it forms a part, and need but a brief statement.

I. The device by which mathematical teaching is to be most promoted is, the author conceives, to be found in the minute subdivision of subjects, and the drill of the student in the separate details before combining them into a whole. The system to which we are thus led is seen in the arrangement of Chapters I., II., and V.

II. By exercises in which the subject is taken up in a concrete form, the formation of mathematical conceptions is greatly facilitated. An application of this principle is seen in the cases where the student is exercised in finding the values of trigonometric functions by construction and measurement.

III. The problems for exercise are quite varied in their character, and are intended to test not only the student's knowledge of the usual methods of computation, but his ability to grasp them and trace them out in the numerous forms they may assume in practical applications.

IV. In the arrangement, strictly logical order has been subordinated to order of teaching. In accordance with this principle, all the simpler applications of the trigonometric functions have been disposed of before their complex relations.

V. The scope of the work is generally limited to the subjects and treatment necessary in the fullest course of mathematics usually taught in our colleges and technological schools. The concluding chapter of each part perhaps exceeds the limit thus set. That of Part I. is an introduction to the employment of imaginary quantities in trigonometric developments, while that of Part II. is an introduction to the higher forms of solid geometry.

VI. To the usual list of subjects treated, has been added a chapter on the theory of polygons. This theory is closely connected with a variety of subjects, including geometry, quaternions, mechanics, graphical statics, surveying, and navigation, and therefore deserves a more prominent place than has hitherto been assigned to it.

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ELEMENTS OF TRIGONOMETRY

PART I.

PLANE TRIGONOMETRY.

CHAPTER I.

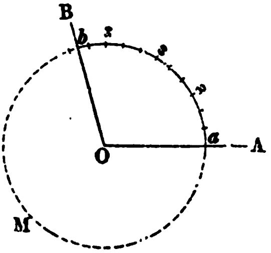
OF GONIOMETRY, OR THE MEASURE OF ANGLES.

1. *Definition.* **Trigonometry** is that branch of geometry in which the relations of lines and angles are treated by algebraic methods.

2. *Def.* An **angle** is the figure formed by two straight lines emanating from the same point, called the **vertex** of the angle.

Def. The lines which form an angle are called its **sides.**

3. *Measures of Angles.* An angle is measured by the length of a circular arc having its centre at the vertex of the angle and its ends on the sides of the angle.



If the angle to be measured is AOB, we conceive that with an arbitrary radius Oa an arc is drawn from a to b.

We regard as the positive direction that in which the arc is described by a motion opposite to that of the hands of a watch, and as the negative direction that in which the hands move.

Hence we may consider the angle as measured either by the arc *ab* considered as positive, or the conjugate arc *aMb* considered as negative. The numerical sum of these two arcs is equal to a circumference.

As an example of the use of algebraic signs, we may mention their application to the latitude of places to distinguish them as *north* and *south*. Thus, a city in 42° north latitude is said to have a latitude of $+42°$, and one 42° south of the equator is said to be in latitude $-42°$.

The absolute length of the arc will depend not only upon the magnitude of the angle, but upon the radius with which the arc is drawn. To avoid ambiguity from this cause, the unit of arc is supposed to be some fixed fraction of the circumference, and therefore greater the greater the radius. The arc is then indicated by the number of units and parts of a unit which it contains, and this number is the same for the same angle whatever the radius may be.

To indicate the angle corresponding to any arc we call it the *angle of the arc*, or, for brevity, the *angle-arc*.

4. *The Sexagesimal Division.* The following is the usual division:

The circumference is divided into 360 units, called degrees;
Each degree is divided into 60 minutes;
Each minute is divided into 60 seconds.

Then

$$1 \text{ circumference} = 360° = 21\,600' = 1\,296\,000'';$$
$$1 \text{ quadrant or right angle} = 90°.$$

This is called the **sexagesimal** division of the circle.

5. *The Centesimal Division.* The sexagesimal division of the circle is by no means so convenient as one in which each unit is 10 times or 100 times greater than the next smaller unit. The centesimal division was introduced by the French geometers at the time of the Revolution. In this system

The circumference is divided into 400 *grades;*
The grade is divided into 100 *minutes;*
The minute is divided into 100 *seconds.*

Hence 1 circumference = 400 grades = 40 000 min. = 4 000 000 sec.; which is commonly written

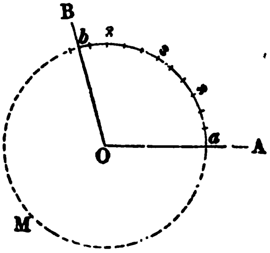
$$400^{\text{gr.}} = 40\,000' = 4\,000\,000''.$$

Notwithstanding its greater convenience, this system never came into general use, owing to the difficulty of changing all the mathematical tables to correspond with it.

6. *Decimals of Degrees or Minutes.* Sometimes, instead of seconds, decimals of a minute are used. Both minutes and seconds may be dispensed with and decimals of a degree be used in their place.

7. *General Measure of an Angle.* The best way of thinking of angular measure is to conceive the side OA of the angle to turn round on O until it reaches the position OB. In thus turning, a point A upon it will describe the circular arc which measures the angle AOB. The length of this arc will then be proportional to the amount by which OB turns in passing from OA to OB.

The side may pass from OA to OB not only by describing the arc ab, but by moving through a whole revolution *plus* the arc ab, or through any number of revolutions *plus* the arc ab. When we consider the angle in the most general way, all these motions will equally measure the angle. Hence we may suppose, indifferently,



Angle AOB = angle-arc ab,

or Angle AOB = angle-arc $ab + 360°$,

or Angle AOB = angle-arc $ab + 720°$,

etc. etc.

If we put C for 360°, or the circumference, the general measure of the angle is

$$\text{Angle } AOB = iC + \text{angle-arc } ab, \quad (1)$$

in which i may be any integer whatever, positive or negative.

We may consider this same form to include the negative measure aMb. For, since $ab + aMb = C$, we have

$$\text{Arc } ab = C - \text{arc } aMb.$$

By substituting this value in (1) it becomes

$$\text{Angle } AOB = (i + 1)C - \text{arc } aMb.$$

Since we have $i = \ldots\ldots -2, -1, 0, 1, 2, 3$, etc., *ad infinitum*, $i + 1$ may go through the same system of integral values as i.

In general, if an angle is n° less than a circumference, we may call it, indifferently,

$$\text{Angle of } (360 - n)^\circ \text{ or angle of } -n^\circ.$$

The general measure of the angle expressed in the form (1) has its most convenient application in Astronomy. The heavenly bodies perform unceasing revolutions, and thus describe continually increasing angles; but each revolution brings them back to what we may consider the same position relative to the centre of motion.

8. In order to give entire algebraic precision to the measure of an angle, we must suppose a distinction between the side *from* which we measure and the side *to* which we measure. In all the preceding examples we have supposed the measure to be *from* OA *to* OB. Had we measured from OB to OA, the arc ab would have been described in the negative direction, or aMb would have been described in the positive direction. Hence we should have had

$$\text{Angle } BOA = -\text{arc } ab \text{ or } +\text{arc } aMb,$$

which is the negative of the corresponding measure from OA to OB. Hence:

By interchanging the sides we change the algebraic sign of the angle.

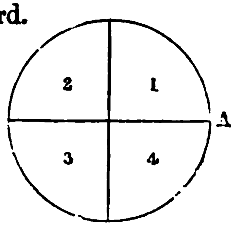
To give uniformity to this mode of measurement, the side OA, from which we measure, is supposed fixed, while the other side varies in direction according to the magnitude of the angle.

When angles are represented in a general way, the side OA may be conceived as extending out horizontally towards the right. Then the other side, OB, will have a definite direction for every angle we choose to assign. For example:

For 90° the side OB will point upward.
" 180° " " " " " to the left.
" 270° " " " " " downward.
" 360° " " " " " to the right.
" 450° " " " " " upward.
etc. etc.

Counting the angles in the negative direction,

For — 90° the side OB will point downward.
" — 180° " " " " " to the left.
" — 270° " " " " " upward.
etc. etc.



9. *Division into Quadrants.* The circle which measures angles is, for convenience, supposed to be divided into quadrants, as in the figure.

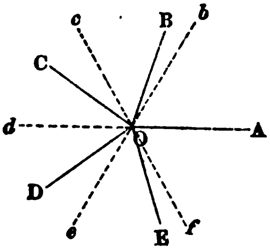
An angle between 0° and 90° is in the first quadrant.
" " " 90° " 180° " " second "
" " " 180° " 270° " " third "
" " " 270° " 360° " " fourth "

Counting the angles negatively from OA,

An angle between 0° and — 90° is in the *fourth* quadrant.
" " " — 90° " — 180° " " *third* "
" " " — 180° " — 270° " " *second* "
" " " — 270° " — 360° " " *first* "

Exercises.

1. From the point O emanate a set of 5 lines making equal angles with each other, and another set of 6 lines making equal angles with each other, the line OA being common to the two sets. Compute the values in degrees of the ten angles AOb, bOB, BOc, etc., to fOA.



2. What is the value of that angle whose negative measure is numerically double its positive measure?

3. If a side starting from the zero point move through — 1905°, in what quadrant will it be found, and what will be the smallest positive measure of the angle?

4. Two arms start together from the same position OA to turn round O, the one going in the positive direction, so as to revolve

in 60 seconds, the other in the negative direction, so as to revolve in 36 seconds. At what angle and in what time will they meet?

5. If two revolving arms start out together from the position 0° in the same direction, the one going 5° a minute and the other 8° a minute, through what arc will each have moved when they again come together? At what angle will they meet? If they continue turning, after how many revolutions of each will they be together at their starting point?

6. Four lines, a, b, c, d, emanate from the same point o, making angle $boc = 2aob$, $cod = 2boc$, $doa = 2cod$. What are the values of the four angles which they form?

7. If an angle of 140° is multiplied successively by -2, -3, -4, -5, -6, -7, in what quadrants will the respective multiples fall, and what will be the smallest positive measures of the several angles formed?

8. Show that the following pairs of angles are supplementary:

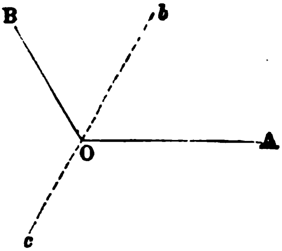
$$90^\circ + x \text{ and } 90^\circ - x;$$
$$270^\circ - x \text{ and } 270^\circ + x;$$
$$60^\circ - x \text{ and } 120^\circ + x.$$

9. Show that the following pairs are complementary:

$$45^\circ - x \text{ and } 45^\circ + x;$$
$$225^\circ - x \text{ and } 225^\circ + x;$$
$$60^\circ - x \text{ and } 30^\circ + x.$$

The Division of Angles.

10. *Bisection.* If the angle $AOB = n^\circ$, and if Ob is its bisector, then $AOb = \frac{1}{2}n^\circ$.



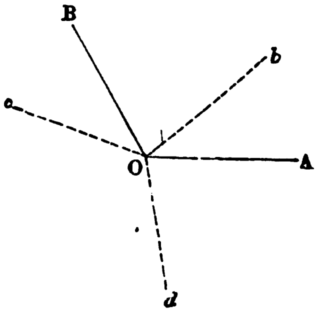
If the side OB revolves about O, and the side Ob also revolves in the same direction half as fast, then Ob will continually bisect the angle AOB.

When OB completes a revolution, returning to the position OB, the bisector Ob will have moved through 180°, and will therefore lie in the opposite direction, Oc. Another revolution of OB will bring the bisector to the position Ob again, yet another to Oc, and so on. Hence:

The general measure of an angle has two bisectors 180° *apart.*

11. *Trisection.* If the side Ob is to continually measure one third the angle AOB as OB revolves, then we must have

$$AOb = \tfrac{1}{3}AOB.$$



If OB, starting from the position in the figure, goes through one revolution, Ob will go through 120° to the position Oc. . A second revolution of OB will bring Ob 120° farther, to Od, and a third to its first position, Ob, after which it will repeat its movements. Hence:

One third the general measure of an angle has three special angular values differing by 120°.

12. *Division into n Parts.* If, as OB revolves, Ob continually measures $\frac{1}{n}$ of it, then every revolution of OB will turn Ob through $\frac{1}{n}$ of a revolution. Hence:

The nth part of the general measure of an angle has n special angular values.

13. *Analytic Deduction.* It will be remarked that, in the preceding sections, what we take the nth part of is not the angle AOB, but the general measure of this angle. This will be clear from the following analytic deduction of the same result.

Let the smallest measure of the angle AOB be α. Then the other measures of this angle (§ 7) will be

$$\alpha + C,\ \alpha + 2C,\ \alpha + 3C\ .\ .\ .\ \alpha + iC.$$

Dividing these quantities by n, the quotients will be

$$\frac{\alpha}{n},\ \frac{\alpha}{n} + \frac{1}{n}C,\ \frac{\alpha}{n} + \frac{2}{n}C,\ \text{etc.} \qquad (1)$$

The $(n+1)$th angle of this series will be

$$\frac{\alpha}{n} + \frac{n}{n}C = \frac{\alpha}{n} + C,$$

which will correspond to the same position of OB as $\frac{a}{n}$ does. The continuation of the series will be

$$\frac{a}{n} + C + \frac{1}{n}C,\ \frac{a}{n} + C + \frac{2}{n}C,\ \text{etc.},$$

showing that the positions will be continually repeated in regular order.

EXERCISES.

1. If $AOB = 30°$, or $30° + C$, or $30° + 2C$, etc., at what angles will $\frac{1}{3}AOB$ fall?

2. How many degrees between the minute-lines on a clock-face?

3. At what angles with XII. are the hour and minute hands of a clock together?

4. If the hour-hand is so displaced that when the minute-hand is at XII. the hour-hand is 2^{m} past XII., at what angles will the hands be together?

5. What values may two thirds of the general measure of an angle of 105° have?

6. If an angle is in the third quadrant, what are the limits between which its bisectors must fall?

7. Between what three sets of limits must a be contained in order that $3a$ may fall in the fourth quadrant?

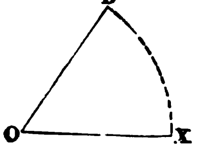
8. Show that while one sixth of the general measure of an angle has six different values, two sixths has only three values, and three sixths only two values. Show that this diminution arises from several values falling together when multiplied by 2 or 3. As an example, take the case when $a = 48°$; $\frac{1}{6}a = 8°$, 68°, etc.

14. *Natural Measure of Angles.* The division of the circumference into 360° is entirely arbitrary, and any other angle than the degree may be taken as the unit.

In purely mathematical investigations, where no division into degrees is required, the length of the radius is taken as the unit of measure.

This unit is called the **radian.**

The radian is therefore the angle subtended by an arc whose length is equal to the radius.



An angle of one radian in which arc XB = radius OX.

To find the relation of this unit of angle to the degree, minute, and second, we note that the ratio of the entire circumference to the diameter is 3.141 592 65, etc. (Geom., Book VI., § 5.)

Hence its ratio to the radius is double this number, or 6.283 185 3, etc. Since the circumference measures 360°, the unit radius will measure $\frac{360^\circ}{6.283\,185\,3}$, or 57°.295 779 5 . . . . Hence

$$\begin{aligned} 1 \text{ radian} &= 57.295\,779\,5 \ldots . \text{ degrees.} \\ &= 3437.746\,77 \ldots . \text{ minutes.} \\ &= 206\,264.806 \ldots . \text{ seconds.} \end{aligned}$$

In mathematics we use the symbol

$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{\frac{1}{2}\text{ circumference}}{\text{radius}} = 3.141\,592\,65 \ldots .$$

Hence, when we take the radian as the unit,

$\frac{1}{2}\pi$ represents an angle of 90°;
π " " " " 180°;
2π " " " " 360° = circumference;
$2n\pi$ " " " " n circumferences.

EXERCISES.

Considering the radius of the circle as unity, what is the length of circular arcs subtending the following angles?

1. 28° 17′ 15″.6. Ans. 0.493 72.
2. 14° 8′ 37″.8 " 0.246 85.
3. 22° 25′ 53″.4 " 0.391 51.
4. 90° " 1.570 80.

NOTE. In these exercises the angle is first to be reduced to a common denomination of measure, either degrees, minutes, or seconds. For instance,

28° 17′ 15″.6 = 101 835″.6 = 1697′.26 = 28°.287 67.

If the radius is 100 metres, how many degrees and minutes will arcs of the following lengths subtend?

5. 100 metres. Ans. 57° 17′ 44″.8.
6. 72 metres. " 41° 15′.1.
7. 310 metres. " 177° 37′.

With what radius will—

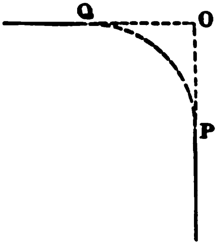
8. Arc of 32 metres' length subtend an angle of 32° ?

Ans. $57^{m}.296$.

9. Arc of 32 metres' length subtend an angle of 32′ ?
10. " " 32 " " " " " " 32″ ?
11. " " 1 metre " " " " " " 67″ ?

Ans. $3078^{m}.6$.

12. Two railways met at right angles at O. They are to be connected by a quadrant PQ, of which the inner rail shall be 600 metres in length. What is the common distance OP and OQ of the switches from the point O in which the two inner rails would meet ? Ans. $381^{m}.97$.



13. In the preceding case, if the rails are 5 feet apart, how much longer will the outer rail of the curve be than the inner one ?

Ans. 7.854 feet.

14. Show that if three circles, equal or unequal, mutually touch each other externally in the points A, B, and C, the sum of the three included arcs $AB + BC + CA$, expressed in angular measure, is equal to a certain constant. What is this constant ?

15. If the two lesser circles, still touching each other, touch the greater one internally, show that the sum of their arcs *minus* the arc of the greater circle, expressed in angular measure, is equal to the same constant as that of the preceding problem.

16. The earth's equatorial diameter being 12,756 kilometres, what is the length of one degree of the equator in kilometres and in miles, assuming 1 metre = 39.37 inches.

17. Explain why a degree of latitude is greater at the poles than at the equator, although the radius of the earth is less.

Remark.—At this stage of his progress, if not sooner, the student should be familiarized with the use of the logarithmic and trigonometric tables, and should employ them in all computations in which they are applicable

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

The Sine, Tangent, and Secant.

15. To investigate the numerical relations between the sides and angles of geometric figures, certain functions of angles are employed in trigonometry. These functions are defined in the following way:

Let OX be that side of the angle from which we measure, the length OX being taken as the radius of a circle. Also, suppose

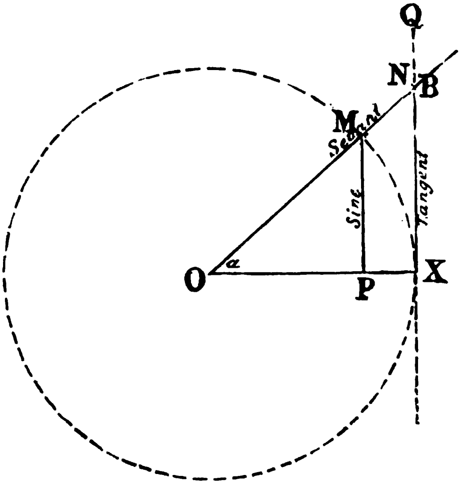
OB, the side to which we measure;

M, the point in which OB intersects the circle;

XQ, the line tangent to the circle at X;

N, the point at which this tangent meets the side OB;

MP, the perpendicular from M upon OX.



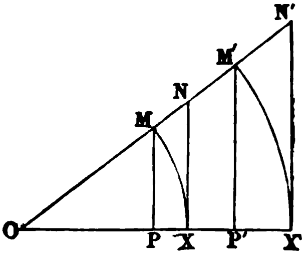
Then taking the radius OX as unity, and expressing other lengths in terms of this unit:

I. The length MP is called the **sine** of the angle XOB.

II. The length NX is called the **tangent** of the angle XOB.

III. The length ON is called the **secant** of the angle XOB.

The absolute lengths of the lines representing the sine, tangent, and secant, considered as lines, will vary with the radius of the circle. This is avoided by taking for the sine, tangent, and secant, not the lines which represent them, but the *ratios* of the lines to the radius of the circle, which ratios will be pure numbers.



We have now to prove that these numbers are the same for the same angle whatever be the radius.

Let $X'ON'$ be the angle.

From the vertex O draw the two arcs XM and $X'M'$ with any two radii OX and OX'.

Erect the respective sines and tangents $PM, XN, P'M', X'N'$.

Then because the triangles OPM, OXN, $OP'M'$, and $OX'N'$ have the angle at O common, and the respective angles at P, X, P', and X' all right angles, and therefore equal, these triangles are equiangular and similar.

Comparing the sides about the equal angles we have the ratio

$$\left.\begin{array}{l} PM : OM = P'M' : OM'; \\ XN : OX = X'N' : OX'; \\ ON : OX = ON' : OX'; \end{array}\right\} \qquad (a)$$

Because $\quad OM = OX =$ radius of inner circle,

and $\quad OM' = OX' =$ radius of outer circle,

we have, by definition,

$$PM : OM = \text{sine of } POM;$$
$$XN : OX = \text{tangent of } POM;$$
$$ON : OX = \text{secant of } POM.$$

The equations (a) now show that the sine, tangent, and secant of the angle will be represented by the same numbers whether we measure them in the inner or outer circle. Therefore:

To each angle of a definite magnitude corresponds
One definite number, called the sine of the angle;
Another definite number, called the tangent of the angle;
Another definite number, called the secant of the angle.

17. *Notation.* If we call a any angle,

Its sine is written .... sin a;
" tangent " .... tan a;
" secant " .... sec a.

NOTE. The representation of the trigonometric functions by lines is for the sake of clearness. They are not really lines, but ratios of lines which are pure numbers. But in studying these numbers the ideas are fixed by representing them by lines, as is done in some departments of algebra. We have only to remember that the lines are not the functions themselves, but lengths proportional to the functions, and therefore admitting of being used to represent the functions. These lengths are, however, really equal to the products of the radius by the corresponding functions. For example, if

$$\sin XOM = \frac{PM}{\text{Radius}},$$

then

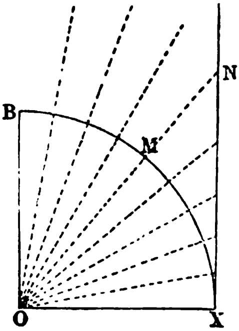
$$\sin XOM \, . \, \text{Radius} = PM.$$

18. *Remark.* The sine of an angle is equal to half the chord of twice the arc of the angle, the radius being supposed unity. Hence:

Any chord in a circle is equal to the radius multiplied by twice the sine of half the angle subtended by the chord.

EXERCISE.

Let the student find by actual measurement with dividers and scale the sine, tangent, and secant of every 10° from 0° to 90° in the following way:



With a radius equal to some unit or some whole number of units on a scale, describe the quadrant XB. Either 4 inches, 5 inches, or a decimetre would be a convenient radius.

Divide the quadrant into 9 arcs of 10° each. Through each point of division, M for instance, draw a radius and continue it until it intersects the tangent at N. Then measure—

1. The distance of each division-point on the arc from the line OX, which distance, divided by the radius, will give the *sine* of the corresponding angle.

2. The distance of each point of intersection, N, from X, which being divided by the radius will give the *tangent* of the angle.

3. The length of each ON, which being divided by the radius will give the *secant* of the corresponding angle.

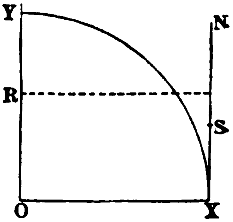
The results should all be expressed in decimals to three places, exhibited in a little table in the following form, and afterward compared with the values found in the trigonometric tables:

Angle.	Sine.	Tangent.	Secant.
0°	. . .	. . .	. . .
10°	. . .	. . .	. . .
20°	. . .	. . .	. . .
30°	. . .	. . .	. . .
etc.	. . .	. . .	. . .

With care the average deviation of the measures from the truth ought not to exceed .005, except in the cases of the tangent and secant of 70° and 80°, which are so great that they cannot be easily found in this way.

19. *To find, by measurement, the angle corresponding to a given sine, tangent, or secant.*

Analysis. If a sine is given, the end of the arc corresponding to the required angle must be at a distance from the line OX equal to the given sine, the radius being unity. Therefore if we take on the perpendicular OY a distance OR equal to the product of the radius by the given sine, and through R draw a parallel to OX, the point in which it intersects the arc will give the required angle.



To find the angle corresponding to a given tangent we take a distance XS equal to the product of the given tangent into the radius. Join OS. The angle XOS will be that required.

For a secant we take the product of the radius by the secant in the dividers, and from O as a centre draw an arc cutting XN in a point N. Joining ON, the angle XON will be that required.

EXERCISES.

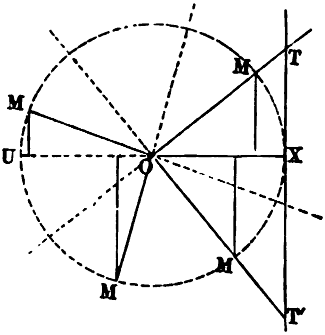
1. Find by measurement the angles of which the sines are $\frac{1}{4}$, 0.3, 0.4, 0.6.

2. Find arc-tan $\frac{1}{2}$, 1, 1.5, 3.

The expression arc-tan is used for brevity to mean the arc corresponding to a given tangent.

3. Find arc-secant 1.5, 2.

20. *Functions of unlimited angles.* Thus far we have considered only the sines, tangents, and secants of angles less than 90°; that is, of angles in the first quadrant (§ 9). As our angle increases to an entire circumference, the functions are determined by the same construction modified to suit the case.



The following are the general definitions:

We first generalize the construction.

On the sides of the angle we take equal lengths OX and OM as the unit of measurement and radius of the circle.

At X we erect a line TXT' perpendicular to OX, extending indefinitely in both directions.

We also suppose the revolving side OM to be produced indefinitely in both directions, and XO to be produced so as to form the diameter XU.

Then, however the side OM may revolve—

I. *The sine of the angle XOM is always represented by the perpendicular from M upon the line OX.* The sine is positive or negative according as M is above or below OX.

II. *The tangent of the angle XOM is always represented by the distance from X to the point in which the side OM produced intersects the line TXT'.* The tangent is positive or negative according as the point of intersection is above or below X.

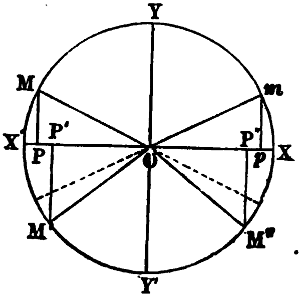
III. *The secant of XOM is the length of OM produced intercepted between O and the vertical line TXT'.* The secant is positive or negative according as it is in the direction from O toward M or in the opposite direction. The positive direction of OM is supposed to revolve with the side OM, and therefore to be always from O toward M.

21. *Changes in the value of the sine.* If we suppose the side Om of the angle XOm to revolve around O, carrying the sine mp with it, the latter will increase to its maximum value, equal to radius unity, when m reaches Y, and angle XOm = 90°. Hence

$$\sin 90° = +1.$$

As m moves from Y to X', the sine will diminish from 1 to zero. Because angle $XOX' = 180°$,

$$\sin 180° = 0.$$



If m passes over X' into the third quadrant, the perpendicular $M'P'$ will be below the line $X'OX$. This change of direction is expressed by changing the algebraic sign of the perpendicular from + to —. This is in accordance with the following general principle:

Whenever distances measured in one direction are considered positive, those in the opposite direction are negative.

Hence also:

In the third quadrant the sine is negative.

When the point m reaches the position Y' it will have moved through three quadrants or 270°, and the sine will coincide with the radius OY' of length unity. Hence

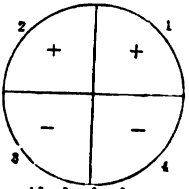
$$\sin 270° = -1. \qquad (1)$$

As m moves from Y' to X, the sine will increase from -1 to 0. Hence:

I. *In the fourth quadrant the sine is negative.*

II. $\sin 360° = \sin 0° = 0.$

The changes of algebraic sign as the angle goes through the four quadrants are shown in the annexed diagram.



Algebraic signs of the sine.

Angles having equal sines. If angle $X'OM = XOm$, the two angles XOm and XOM will be supplementary. Also in this case the triangles XOm and $X'OM$ will be identically equal; so that

$$PM = pm.$$

Now PM represents by construction the sine of XOM, and pm the sine of XOm. Hence:

The sines of supplementary angles are equal.

In symbolic language this theorem is expressed thus: If α be any angle, then

$$\sin(180^\circ - \alpha) = \sin \alpha \tag{2}$$

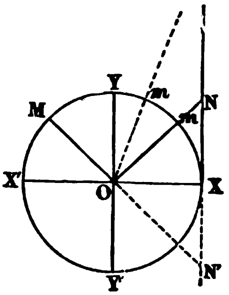
and

$$\sin(90^\circ + \alpha) = \sin(90^\circ - \alpha).$$

If the points M' and M'' are equally distant from Y', so that angle $M'OY'$ = angle $Y'OM''$, which angle call y, the sines $P'M'$ and $P''M''$ will be equal. Hence, whatever be y,

$$\sin(270^\circ - y) = \sin(270^\circ + y).$$

22. *Changes in the tangent.* As the line Om revolves round O and m approaches Y, the point of intersection N will move upwards without limit. As m reaches Y, Om will become parallel to the tangent line, and the point N will recede to infinity. Hence:



The tangent of 90° *is infinite.*

When m is in the second quadrant, suppose in the position M, the revolving side OM will not intersect the tangent line at all in the positive direction OM. We must therefore suppose the revolving line to be produced in the negative direction ON' so as to intersect the tangent line at N' below X. The distance XN' is then to be regarded as negative. Hence:

In the second quadrant the tangent is negative.

Following the motion, we see that when m reaches X', N' reaches X and the tangent becomes zero. Hence

$$\tan 180^\circ = 0.$$

When m is in the third quadrant, N passes above X and the tangent is positive, so that

In the third quadrant the tangent is positive.

Continuing the reasoning, we see that the tangent of 270° is infinite, and that in the fourth quadrant the tangent is negative.

23 *Changes in the secant.* The secant is defined as the distance ON from O to the point N, in which the revolving side intersects the tangent line XN.

When m falls on X and the angle is zero, the secant is equal to $0X$, or unity. Hence

$$\sec 0° = 1. \tag{3}$$

As m moves from X to Y, the secant increases without limit and becomes infinite when m reaches Y. Hence

$$\sec 90° = \infty. \tag{4}$$

As m moves from Y through X' to Y', the intersection of the revolving line with the tangent line falls in the negative part of OM, or in the direction ON'. Hence:

In the second and third quadrants the secant is negative.

At Y', when the angle is 270°, the secant again becomes infinite.

Between Y' and X, or in the fourth quadrant, it is again positive.

24. If we suppose the revolving line to make an integral number of revolutions from any point, it will return to its original position, and all the trigonometric functions will have the values corresponding to that position. Hence, if C is a circumference and n any integer,

$$\left.\begin{aligned}\sin(nC+\alpha) &= \sin\alpha;\\ \tan(nC+\alpha) &= \tan\alpha;\\ \sec(nC+\alpha) &= \sec\alpha.\end{aligned}\right\} \tag{5}$$

In other words,

The values of the trigonometric functions are not altered by increasing the angle by any integral number of circumferences.

If the angle is increased indefinitely, the values of these functions continually repeat themselves. This fact is expressed by saying that these functions are *periodic.*

EXERCISES.

Prove the following expressions for the trigonometrical functions of angles of more than 90° by the necessary diagrams. The angle x may be supposed less than 90°, though this restriction is not necessary to the validity of the formulæ.

$$\left.\begin{aligned}
&1.\quad \sin(90^\circ + x) = \sin(90^\circ - x);\\
&2.\quad \sin(180^\circ + x) = -\sin x;\\
&3.\quad \sin(270^\circ + x) = -\sin(90^\circ - x).
\end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned}
&4.\quad \tan(90^\circ + x) = -(\tan 90^\circ - x);\\
&5.\quad \tan(180^\circ + x) = \tan x;\\
&6.\quad \tan(270^\circ + x) = -(\tan 270^\circ - x).
\end{aligned}\right\} \qquad (7)$$

$$\left.\begin{aligned}
&7.\quad \sec(90^\circ + x) = -\sec(90^\circ - x);\\
&8.\quad \sec(180^\circ + x) = -\sec x;\\
&9.\quad \sec(270^\circ + x) = -\sec(270^\circ - x).
\end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned}
&10.\quad \sin(-x) = -\sin x;\\
&11.\quad \tan(-x) = -\tan x;\\
&12.\quad \sec(-x) = \sec x.
\end{aligned}\right\} \qquad (9)$$

NOTE. When we have the values of the trigonometric functions from 0 to 90°, we can by these formulæ find the values for all angles.

The Cosine, Cotangent, and Cosecant.

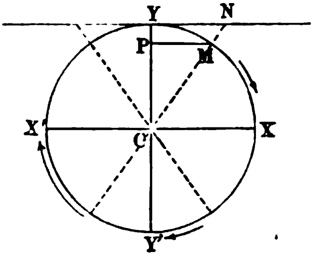
25. In the preceding sections we have supposed the side of the angle from which we count the degrees to go out toward the right, and the positive direction of motion to be opposite to that of the hands of a watch. But this restriction is only to fix the thought. We may suppose the angle to have any situation and to be counted in any direction without changing the values of the sine, tangent, and secant, provided that we reckon the lengths of the lines representing the functions in the right way.

Let us count the angle from OY in the direction toward OX. The tangent line must then touch the circle at Y, and its positive direction must be toward the right.

Then (radius $OY = 1$) the lines

$$\left.\begin{aligned}
PM &= \sin POM,\\
YN &= \tan POM,\\
ON &= \sec POM,
\end{aligned}\right\} \qquad (a)$$

will have the same values as in an angle equal to *POM* counted in the usual way from *OX* toward *Y*.



Moreover, the changes of sign will be the same as before through the whole circle, namely:

From *X* to *Y'* (now the second quadrant) the sine *PM* will be positive because it is measured to the right.

From *Y'* to *X'* (the third quadrant) it will be negative because it is measured toward the left.

It will also be negative from *X'* to *Y* (now the fourth quadrant).

The corresponding propositions can be shown for the tangent and secant.

Now because angle $XOM +$ angle $MOY =$ angle $XOY = 90°$, *MOY* is the *complement* of *XOM*. Therefore the equations (*a*) may be written

$$PM = \text{sin comp. of } XOM;$$
$$YN = \text{tan comp. of } XOM;$$
$$ON = \text{sec comp. of } XOM.$$

Because when we have an angle its complement can always be determined by subtracting it from 90°, we can always find the sine, tangent, and secant of the complement when we know the angle. Therefore the sine, tangent, and secant of the complement of an angle may be regarded as three additional trigonometrical functions of the angle itself. They are named thus:

The sine of the complement is called the **cosine** of the angle.

The tangent of the complement is called the **cotangent** of the angle.

The secant of the complement is called the **cosecant** of the angle.

Thus the new functions are defined in the following way:

$$\left.\begin{aligned}\text{cosine } \alpha &= \sin(90° - \alpha);\\ \text{cotang } \alpha &= \tan(90° - \alpha);\\ \text{cosecant } \alpha &= \sec(90° - \alpha).\end{aligned}\right\} \qquad (10)$$

The words cosine, cotangent, and cosecant are abbreviated to cos, cot, cosec, respectively.

The forms (10) enable us to find the cosine, cotangent, and cosecant of an angle when we know the sine, tangent, and secant of its complement. Thus if the cosine of 60° is required, we have

$$\cos 60^\circ = \sin (90^\circ - 60^\circ) = \sin 30^\circ.$$

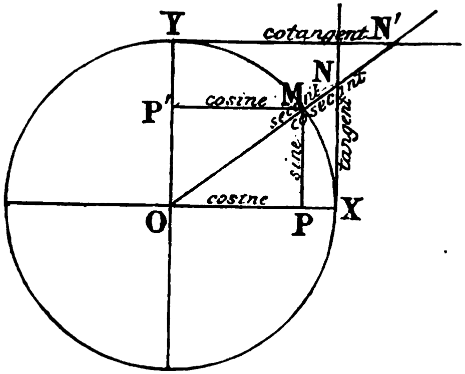
Also, by substituting $90^\circ - \alpha$ for α, we find

$$\left.\begin{aligned} \sin \alpha &= \cos (90^\circ - \alpha); \\ \tan \alpha &= \cot (90^\circ - \alpha); \\ \sec \alpha &= \operatorname{cosec} (90^\circ - \alpha). \end{aligned}\right\} \qquad (11)$$

The versed-sine and co-versed-sine. Besides these six functions, two others, the *versed-sine* and *co-versed-sine*, are sometimes used. Their definitions are:

$$\text{Versed-sine} = PX = 1 - \text{cosine};$$
$$\text{Co-versed-sine} = OY - PM = 1 - \text{sine}.$$

26. The six trigonometrical functions may be represented on a single diagram. The functions as written are all those of the angle XOM. For the secant and cosecant we have



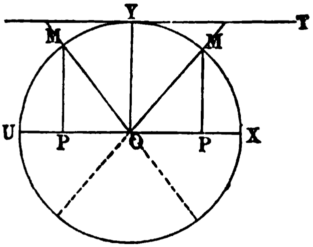
$$ON = \sec XOM,$$
$$ON' = \operatorname{cosec} XOM,$$

because XOM is the complement of MOY.

Because $PM \parallel OP'$ and $OP \parallel P'M$, therefore $OP = P'M$, so that we may take either OP or $P'M$ as the cosine of XOM.

27. The general definitions of § 20 may be extended to complementary functions, thus:

Having $OY \perp OX$ and $OY = OX = 1$, we draw through Y an indefinite line YT parallel to OX. Then—

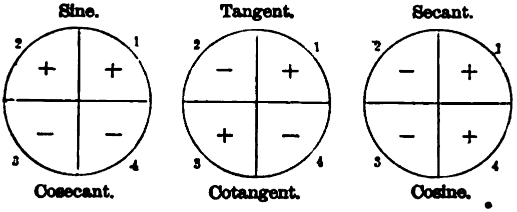


IV. *The cosine of any angle XOM is represented by the distance OP to the foot of the sine, positive or negative according to its direction.*

V. *The cotangent of any angle XOM is represented by the length from Y to the point in which OM produced intersects YT.*

VI. *The cosecant of XOM is represented by the length of the side OM, intercepted between O and the line YT.*

The algebraic signs of the several functions in the four quadrants are shown in the following diagram.



28. The following are the limiting values of the trigonometrical functions:

I. *Sine and cosine.* The sine MP and cosine OP are necessarily not greater in absolute value than $OM = 1$. The limits of these functions are therefore $+1$ and -1.

II. *Secant and cosecant.* Since the tangent line lies without the circle, a secant can never be less than unity in absolute magnitude. But we have found that it may increase to infinity in either the positive or negative direction. Hence the limits of the secant and cosecant are 1 and infinity, and -1 and $-\infty$.

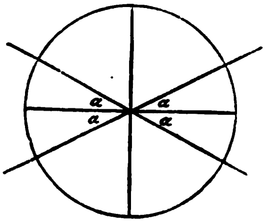
III. *Tangent and cotangent.* The limits of the tangent are easily seen to be $-\infty$ and $+\infty$, or a tangent and cotangent may have any value whatever.

29. When we know the numerical values of the sine, tangent, and secant of all angles from 0° to 90°, we have the values of all six functions of any angle whatever, because as we go around the circle the values of the functions are simply repetitions of the values between 0° and 90°.

Let α be any angle less than 90°. Then any angle in the first quadrant may be represented by α.

In the second quadrant it may be represented by $180° - \alpha$.
" third " " " " $180° + \alpha$.
" fourth " " " " $360° - \alpha$.

The student should now have no difficulty in demonstrating the following relations by completing the construction indicated in the margin and attending to the general definitions §§ 20, 27. Some of these relations have already been given or explained.



Second Quadrant.

$$\left.\begin{aligned}
\sin(180^\circ-\alpha) &= \sin\alpha;\\
\cos(180^\circ-\alpha) &= -\cos\alpha = -\sin(90^\circ-\alpha);\\
\tan(180^\circ-\alpha) &= -\tan\alpha;\\
\cot(180^\circ-\alpha) &= -\cot\alpha = -\tan(90^\circ-\alpha);\\
\sec(180^\circ-\alpha) &= -\sec\alpha;\\
\operatorname{cosec}(180^\circ-\alpha) &= \operatorname{cosec}\alpha = \sec(90^\circ-\alpha).
\end{aligned}\right\}\quad(12)$$

Third Quadrant.

$$\left.\begin{aligned}
\sin(180^\circ+\alpha) &= -\sin\alpha;\\
\cos(180^\circ+\alpha) &= -\cos\alpha = -\sin(90^\circ-\alpha);\\
\tan(180^\circ+\alpha) &= \tan\alpha;\\
\cot(180^\circ+\alpha) &= \cot\alpha = \tan(90^\circ-\alpha);\\
\sec(180^\circ+\alpha) &= -\sec\alpha;\\
\operatorname{cosec}(180^\circ+\alpha) &= -\operatorname{cosec}\alpha = -\sec(90^\circ-\alpha).
\end{aligned}\right\}\quad(13)$$

Fourth Quadrant.

$$\left.\begin{aligned}
\sin(360^\circ-\alpha) &= -\sin\alpha;\\
\cos(360^\circ-\alpha) &= \cos\alpha = \sin(90^\circ-\alpha);\\
\tan(360^\circ-\alpha) &= -\tan\alpha;\\
\cot(360^\circ-\alpha) &= -\cot\alpha = -\tan(90^\circ-\alpha);\\
\sec(360^\circ-\alpha) &= \sec\alpha;\\
\operatorname{cosec}(360^\circ-\alpha) &= -\operatorname{cosec}\alpha = -\sec(90^\circ-\alpha).
\end{aligned}\right\}\quad(14)$$

We may equally express the six functions of all angles in terms of the six functions of angles not greater than 45°. Let γ represent any angle not greater than 45°. We may then represent

Any angle from 0° to 45° by γ;
" " 45° " 90° " $90^\circ-\gamma$;
" " 90° " 135° " $90^\circ+\gamma$;

Any angle from 135° to 180° by $180° - \gamma$;
“ “ 180° “ 225° “ $180° + \gamma$;
“ “ 225° “ 270° “ $270° - \gamma$;
“ “ 270° “ 315° “ $270° + \gamma$;
“ “ 315° “ 360° “ $360° - \gamma$.

Then, in addition to the relations (12), (13), and (14), which will remain true when we write γ instead of α, we shall have the following, which the student should prove.

To do this let the student suppose that in the diagram § 26 angle $XOM = \gamma$, and let him construct the six functions for angles of $90° + \gamma$, $270° - \gamma$, etc., and compare the lines representing them with the lines on the diagram of § 26.

The set corresponding to the first quadrant are already given in (10) and (11).

Second Quadrant.

$$\left.\begin{aligned}
\sin(90° + \gamma) &= \cos\gamma;\\
\cos(90° + \gamma) &= -\sin\gamma;\\
\tan(90° + \gamma) &= -\cot\gamma;\\
\cot(90° + \gamma) &= -\tan\gamma;\\
\sec(90° + \gamma) &= -\operatorname{cosec}\gamma;\\
\operatorname{cosec}(90° + \gamma) &= \sec\gamma.
\end{aligned}\right\} \qquad (15).$$

Third Quadrant.

$$\left.\begin{aligned}
\sin(270° - \gamma) &= -\cos\gamma;\\
\cos(270° - \gamma) &= -\sin\gamma;\\
\tan(270° - \gamma) &= \cot\gamma;\\
\cot(270° - \gamma) &= \tan\gamma;\\
\sec(270° - \gamma) &= -\operatorname{cosec}\gamma;\\
\operatorname{cosec}(270° - \gamma) &= -\sec\gamma.
\end{aligned}\right\} \qquad (16).$$

Fourth Quadrant.

$$\left.\begin{aligned}
\sin(270° + \gamma) &= -\cos\gamma;\\
\cos(270° + \gamma) &= \sin\gamma;\\
\tan(270° + \gamma) &= -\cot\gamma;\\
\cot(270° + \gamma) &= -\tan\gamma;\\
\sec(270° + \gamma) &= \operatorname{cosec}\gamma;\\
\operatorname{cosec}(270° + \gamma) &= -\sec\gamma.
\end{aligned}\right\} \qquad (17).$$

Among the preceding forms of this chapter, the following are of especially frequent application:

$$\left.\begin{array}{l}\sin\alpha = \cos(90^\circ - \alpha) = \cos(\alpha - 90^\circ) = -\cos(\alpha + 90^\circ);\\ \cos\alpha = \sin(90^\circ - \alpha) = -\sin(\alpha - 90^\circ) = +\sin(\alpha + 90^\circ).\end{array}\right\} (18)$$

Exercises.

1. Express the six functions of the following angles in terms of the three functions sine, tangent, and secant of angles less than 90°:

sin 105°; 200°; 295°.
cos 105°; 200°; 295°.
tan 105°; 200°; 295°.
cot 105°; 200°; 295°.
sec 105°; 200°; 295°.
cosec 105°; 200°; 295°.

2. The following table shows the values of four of the functions for every 10° of the first 40° to two places of decimals. By means of these values extend the table to 360°, showing the values of all four functions for each angle:

Angle.	Sin.	Tan.	Cot.	Cos.
0°	0·00	0·00	∞	+1·00
10°	+0·17	+0·18	+5·67	+0·98
20°	+0·34	+0·36	+2·75	+0·94
30°	+0·50	+0·58	+1·73	+0·87
40°	+0·64	+0·84	+1·19	+0·77
50°				
60°				
70°				
etc., to 360°				

3. Demonstrate the relations (18) by drawing a diagram showing an arbitrary angle α and an angle 90° greater and less, with the lines representing the sines and cosines.

4. What relations subsist between the following pairs of functions?

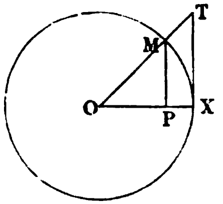
(a) $\sin(45^\circ + a)$ and $\cos(45^\circ - a)$;
(b) $\sin(135^\circ + a)$ and $\cos(135^\circ - a)$;
(c) $\tan(225^\circ + a)$ and $\cot(225^\circ - a)$;
(d) $\sec(315^\circ + a)$ and $\operatorname{cosec}(315^\circ - a)$.

30. *Special values of trigonometric functions.* If angle $XOM = 45°$, we shall also have $OMP = 45°$, and therefore

$$OP^2 + PM^2 = 2PM^2 = OM^2,$$

whence $$PM = OM\sqrt{\tfrac{1}{2}}.$$

Therefore $\sin XOM = \sin 45° = \sqrt{\tfrac{1}{2}}$. (*a*)



In the same way we find

$$XT = OX,$$

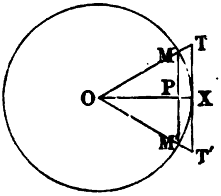
whence $\tan XOM = \tan 45° = 1$. (*b*)

Again, $$OT = \sqrt{OX^2 + XT^2} = \sqrt{2}\,OX;$$

whence $\sec XOM = \sec 45° = \sqrt{2}$. (*c*)

Next, let angle $XOM = 30°$.

Make angle $XOM' = XOM = 30°$.



The triangles $M'OM'$ and TOT' then have each of their angles 60°, and so are equilateral. Therefore $MP = \tfrac{1}{2}MM' = \tfrac{1}{2}OM = \tfrac{1}{2}OX$.

Hence $\sin 30° = \tfrac{1}{2}$. (*d*)

In the same way

$$XT = \tfrac{1}{2}OT,$$

$$OT^2 - XT^2 = OX^2 = 1;$$

or $$\tfrac{3}{4}OT^2 = 1, \qquad OT = \sqrt{\tfrac{4}{3}} = \frac{2}{\sqrt{3}}, \qquad XT = \frac{1}{\sqrt{3}}.$$

Also, $$OP = \sqrt{OP^2 - MP^2} = \sqrt{1 - \tfrac{1}{4}} = \frac{\sqrt{3}}{2}.$$

Hence $$\tan 30° = \frac{1}{\sqrt{3}}, \qquad (e)$$

$$\sec 30° = \frac{2}{\sqrt{3}}, \qquad (f)$$

$$\cos 30° = \frac{\sqrt{3}}{2}. \qquad (g)$$

Functions of 18°. It is shown in geometry that if the radius of a circle be divided in extreme and mean ratio, the greater segment will be the chord of 36°; that is, twice the sine of 18°.

Putting 1 for the radius and r for the greater segment, the condition that the division shall be in extreme and mean ratio is

$$1 : r :: r : 1 - r,$$

or, equating the product of the means to that of the extremes,

$$r^2 = 1 - r.$$

The solution of this quadratic equation gives

$$r = \frac{-1 \pm \sqrt{5}}{2}.$$

The positive root is the only one we want. Hence

$$\sin 18^\circ = \frac{\sqrt{5} - 1}{4}.$$

We then find

$$\cos^2 18^\circ = \text{radius}^2 - \sin^2 18^\circ = 1 - \sin^2 18^\circ.$$

Hence

$$\cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$

31. Angles corresponding to given trigonometric functions.

When the value of a trigonometric function is given and the angle is required, there are always two solutions to the problem.

The Sine. It has already been shown that two supplementary angles have the same sine. Hence if α is an angle corresponding to a given sine, $180^\circ - \alpha$ will be another angle equally corresponding to it.

We may also say that if $90^\circ - \beta$ be an angle corresponding to any sine, $90^\circ + \beta$ will also correspond to it, because the sum of these two angles is 180°. The same statement applies to the angles $270^\circ - \beta$ and $270^\circ + \beta$.

Unless there is some restriction upon the angle to be chosen, we cannot decide which angle to take. The most common restriction is that the angle must be between the limits -90° and $+90^\circ$, or must be in either the first or fourth quadrant. There will then be between these limits one angle and only one for a given sine. Since the measurements of latitude on the surface of the earth are restricted between limits -90° and $+90^\circ$, the latitude of a place is completely fixed by its sine.

The Cosine. The construction of the cosine shows that it has equal values for positive and negative angles. Hence if α be an angle corresponding to a given cosine, $-\alpha$ or $360^\circ - \alpha$ will equally correspond to it.

Hence when an angle is determined by its cosine, either of these two angles may be chosen unless some restriction is placed upon the choice. The most common restriction is that the angle shall be positive and less than 180°; that is, in the first or second quadrant. For every cosine there will be one and only one angle between the limits 0° and 180°.

The Tangent. Since two angles which differ by 180° have the same tangent, it follows that if α be an angle corresponding to any tangent, $180° + \alpha$ will equally correspond to it. Hence there is always a choice between these two angles, unless some restriction is placed upon the angle.

The Cotangent. The cotangent being determined, like the tangent, by the intersection of the revolving side with a tangent line, every pair of angles corresponding to the same tangent will also correspond to the same cotangent. Hence α and $180° + \alpha$ always have the same cotangent.

The Secant. We readily see that the angles α and $-\alpha$ or $360° - \alpha$ have the same secant. Hence if α be an angle corresponding to a given secant, $360° - \alpha$ will be another angle corresponding to that same secant.

The Cosecant. From the diagram (§ 27) it is easy to show that any two supplementary angles have the same cosecant.

EXERCISES.

What other angles have the same sines as the following?

1. 105°; 2. 185°; 3. 290°.

What other angles have the same cosines as the following?

4. 72°; 5. 165°; 6. 320°.

What other angles have the same tangents as the following?

7. 50°; 8. 205°; 9. 355°.

10. A pair of angles having the same sine differ by 24°. What angles are they?

11. A pair of angles having the same cosine differ by 110°. What angles are they?

NOTE. There are two pairs of angles which answer each of the two last questions.

Find two values of α from each of the following equations:

12. $\sin\alpha = -\cos 23^\circ$. Ans. $\alpha = 247^\circ$ or 293°.
13. $\cos\alpha = -\sin 23^\circ$.
14. $\sin\alpha = \cos\beta$. Ans. $\alpha = 90^\circ - \beta$ or $90^\circ + \beta$.
15. $\tan\alpha = \cot 72^\circ$.
16. $\cot\alpha = -\tan 175^\circ$.
17. $\tan\alpha = -\cot\beta$.
18. $\sec\alpha = \operatorname{cosec} 32^\circ$.
19. $\operatorname{cosec}\alpha = -\sec 32^\circ$.
20. $\sec\alpha = \operatorname{cosec}\beta$.
21. $\sin\alpha = \sin 23^\circ$.
22. $\sin\alpha = \cos(90^\circ - x)$. Ans. $\alpha = x$ or $180^\circ - x$.
23. $\sin\alpha = \cos(90^\circ + x)$. Ans. $\alpha = -x$ or $180^\circ + x$.
24. $\sin\alpha = \cos(270^\circ + x)$.
25. $\sin\alpha = \cos(270^\circ - x)$.
26. $\cos\alpha = \cos(180^\circ - x)$.
27. $\cos\alpha = \cos(180^\circ + x)$.

32. *Extension to unlimited angles.* Since when any integral number of circumferences is added to an angle its trigonometric functions remain unaltered, we must, to find the most general expression for the angle corresponding to a given function, add an arbitrary number of circumferences to the angles found in the last section. Then the most general expression for angles which have the same sine will be

$$nC + 90^\circ + \alpha \qquad \text{and} \qquad nC + 90^\circ - \alpha,$$

in which n may take all integral values, positive and negative, including zero, and α must have such a value that $90 - \alpha$ shall correspond to the given sine.

This statement also means that all the angles formed by giving different values to n, while α remains constant, will have the same sine.

Example. Let us suppose the given sine to be that of 65°. Then $\alpha = 25^\circ$, and the pairs of angles

$$\begin{array}{cccc}
65^\circ, & 115^\circ; & -C + 65^\circ, & -C + 115^\circ; \\
C + 65^\circ, & C + 115^\circ; & -2C + 65^\circ, & -2C + 115^\circ; \\
2C + 65^\circ, & 2C + 115^\circ; & \text{etc.} & \text{etc.,} \\
\text{etc.} & \text{etc.;} & &
\end{array}$$

will all have this same sine.

EXERCISES.

1. Having given

$$\sin 2x = \tfrac{1}{2},$$

find the four corresponding values of x within the first circumference. Note that $\sin 30° = \tfrac{1}{2}$. *Ans.* 15°, 75°, 195°, 255°.

2. Having given

$$\sin 2\alpha = \sqrt{\tfrac{1}{2}},$$

find the four corresponding values of α.

First find the two values of 2α, and then, by §§ 10–13, the two values of half of each of these angles.

3. Having given

$$\cos 2\beta = \tfrac{1}{2},$$

find the four corresponding values of β.

4. Having given

$$\cos 2\beta = -\tfrac{1}{2},$$

find the four corresponding values of β.

5. If $\tan 2\alpha = -\sqrt{3}$, find four values of α.

6. Show that if the value of $\tan \tfrac{1}{2}\alpha$ be given there will be only one value of α to correspond to it within the first circumference.

7. If $\sin \tfrac{1}{2}x = \sin 15°$,

find what two values may x have, and show that these values will have the same cosine.

8. Form the most general expression for all angles having the same cosine.

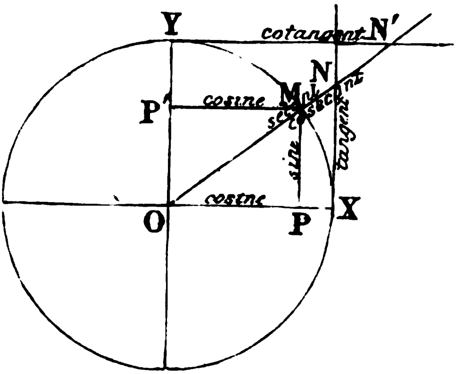
9. Form the most general expression for all angles having the same tangent.

Relations between the Six Trigonometric Functions.

33. When we know any one trigonometric function of an angle two definite values of the angle can then be determined, by constructions like those of § 19. Knowing the angle, the values of the other functions can be found. Hence from any one function of an angle all the others can be found. We have now to investigate the algebraic relations by which this may be done, seeking to express each function in terms of the five others.

Let us put

$$\alpha = \text{angle } XOM.$$



Because OPM is a right-angled triangle,

$$OP^2 + PM^2 = OM^2,$$

or

$$\left(\frac{OP}{OM}\right)^2 + \left(\frac{PM}{OM}\right)^2 = 1.$$

Hence

$$\cos^2 \alpha + \sin^2 \alpha = 1,$$

which gives

$$\cos \alpha = \pm \sqrt{1 - \sin^2 \alpha}. \quad (1)$$

The significance of the double sign ± is this: whenever a sine is given there will be two angles to correspond to it, of which one will be as much less than 90° as the other is greater. The cosines of these angles will be equal with opposite algebraic signs.

The similar triangles OPM and OXN give

$$OP : PM = OX : XN;$$
$$OP : OM = OX : ON.$$

Substituting for the lines their algebraic equivalents, the first proportion gives

$$\cos \alpha : \sin \alpha = 1 : \tan \alpha.$$

Hence $$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\sin \alpha}{\sqrt{1 - \sin^2 \alpha}}, \quad (2)$$

which gives the tangent in terms of the sine and cosine.

The second proportion gives

$$\cos \alpha : 1 = 1 : \sec \alpha,$$

whence $$\sec \alpha = \frac{1}{\cos \alpha} = \frac{1}{\sqrt{1 - \sin^2 \alpha}}. \quad (3)$$

We conclude from this:

The product of the cosine and secant is equal to unity.

In other words, *the secant and cosine are reciprocals of each other.*

By a similar course of reasoning upon the complementary triangle we find that *the cosecant and sine are reciprocals of each other.*

The similar triangles $OP'M$ and OYN' give

$$OP' : P'M = OY : YN';$$

but $$OP' = PM = \sin\alpha,$$

whence $$\sin\alpha : \cos\alpha = 1 : \cot\alpha,$$

and $$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}. \tag{4}$$

Comparing with (2) we have the relation: *The product of the tangent and cotangent of any angle is unity.*

In other words, *the tangent and cotangent are reciprocals of each other.*

We thus reach the general conclusion *that the three complementary functions are each the reciprocal of one of the three other functions*, namely:

$$\text{cosine} = \frac{1}{\text{secant}};$$

$$\text{cotangent} = \frac{1}{\text{tangent}};$$

$$\text{cosecant} = \frac{1}{\text{sine}}.$$

EXERCISES.

1. If $\sin\gamma = 0.60$, find $\cos\gamma$, $\tan\gamma$, $\cot\gamma$, $\sec\gamma$, and $\operatorname{cosec}\gamma$
2. Find the values of the same functions when $\cos\gamma = 0.60$.
3. Prove that the mean proportional between $a\cos x$ and $a\sec x$ is a.
4. Prove that the mean proportional between $a\tan\beta$ and $b\cot\beta$ is $\sqrt{ab}$.

34. *Expression of each function in terms of the others.* By means of the relations (1) to (4) any one trigonometric function may be expressed in terms of any other by algebraic substitutions.

The following are the expressions which we thus obtain. All not already given should be deduced by the student as an exercise.

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+\cot^2\alpha}}$$
$$= \frac{\sqrt{\sec^2\alpha - 1}}{\sec\alpha} = \frac{1}{\operatorname{cosec}\alpha}.$$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}$$

$$= \frac{1}{\sec\alpha} = \frac{\sqrt{\operatorname{cosec}^2\alpha-1}}{\operatorname{cosec}\alpha}.$$

$$\tan\alpha = \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}} = \frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha} = \frac{1}{\cot\alpha}$$

$$= \sqrt{\sec^2\alpha-1} = \frac{1}{\sqrt{\operatorname{cosec}^2\alpha-1}}$$

$$\cot\alpha = \frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha} = \frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}} = \frac{1}{\tan\alpha}$$

$$= \frac{1}{\sqrt{\sec^2\alpha-1}} = \sqrt{\operatorname{cosec}^2\alpha-1}.$$

$$\sec\alpha = \frac{1}{\sqrt{1-\sin^2\alpha}} = \frac{1}{\cos\alpha} = \sqrt{1+\tan^2\alpha}$$

$$= \frac{\sqrt{1+\cot^2\alpha}}{\cot\alpha} = \frac{\operatorname{cosec}\alpha}{\sqrt{\operatorname{cosec}^2\alpha-1}}.$$

$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha} = \frac{1}{\sqrt{1-\cos^2\alpha}} = \frac{\sqrt{1+\tan^2\alpha}}{\tan\alpha}$$

$$= \sqrt{1+\cot^2\alpha} = \frac{\sec\alpha}{\sqrt{\sec^2\alpha-1}}.$$

NOTE. In algebraic work of this sort the student will find it convenient, instead of writing sin α, cos α, etc., in full, to use a single symbol for each function; for example, he may put:

$$s = \sin\alpha; \quad \frac{1}{s} = \operatorname{cosec}\alpha.$$

$$t = \tan\alpha; \quad \frac{1}{t} = \cot\alpha.$$

$$c = \cos\alpha; \quad \frac{1}{c} = \sec\alpha.$$

It will be seen that the second members of most of these equations are surds, showing that their values may be either positive or negative. The result may be expressed thus:

Whenever one trigonometric function is given, the four other functions which are not its reciprocal may have either of two equal values with contrary signs.

This arises from the fact that every such function may belong to either of two angles, and affords an example of the correspondence between geometric and algebraic results.

Exercises.

1. From the special values of sin 30° and sin 45° found in § 30, namely,

$$\sin 30^\circ = \tfrac{1}{2},$$
$$\sin 45^\circ = \sqrt{\tfrac{1}{2}},$$

find the values of the other five functions of 30° and of 45°.

Ans. $\cos 30^\circ = \frac{\sqrt{3}}{2}$; $\cos 45^\circ = \sqrt{\tfrac{1}{2}}$.

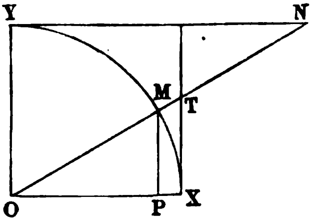
$\tan 30^\circ = \frac{1}{\sqrt{3}}$; $\tan 45^\circ = 1$.

$\cot 30^\circ = \sqrt{3}$; $\cot 45^\circ = 1$.

$\sec 30^\circ = \frac{2}{\sqrt{3}}$; $\sec 45^\circ = \sqrt{2}$.

$\operatorname{cosec} 30^\circ = 2$; $\operatorname{cosec} 45^\circ = \sqrt{2}$.

2. It has been shown that the three products *sin* × *cosec*, *tan* × *cot*, and *cos* × *sec* are each unity. When we replace the functions by the lines which represent them, the products represent the areas of the rectangles contained by the lines, and unity is replaced by the square of the radius. Hence we have the following theorems, which are to be proved by the similar triangles in the accompanying figure, where the construction is that of the trigonometric functions.



I. Rectangle $XT.YN = OX^2$, corresp. to tan × cot = 1.

II. Rectangle $MP.ON = OX^2$, " " sin × cosec = 1.

III. Rectangle $OP.OT = OX^2$, " " cos × sec = 1.

3. From the value of sin 18° in § 30 find the sides of the regular inscribed and circumscribed decagons of a circle of radius *a*.

Prove the following relations:

4. $$1 + \sin x = \frac{\cos^2 x}{1 - \sin x}.$$

5. $$\frac{\sec\theta + \operatorname{cosec}\theta}{\sec\theta - \operatorname{cosec}\theta} = \frac{1 + \cot\theta}{1 - \cot\theta} = \frac{\tan\theta + 1}{\tan\theta - 1}.$$

6. $$\tan\theta + \cot\theta = \frac{\sec^2\theta + \operatorname{cosec}^2\theta}{\sec\theta \operatorname{cosec}\theta}.$$

7. $$\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}.$$

8. $$\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta.$$

9. $$\frac{\tan\theta - \sin\theta}{\sin^2\theta} = \frac{\sec\theta}{1 + \cos\theta}.$$

10. What angle is that of which the tangent is double the sine?

11. What must be the value of the cosine in order that the tangent may be n times the sine? Ans. $\frac{1}{n}$.

12. Prove $(r\cos x)^2 + (r\sin x\sin u)^2 + (r\sin x\cos u)^2 = r^2$.

13. Prove $(a\sin\gamma)^2 + (a\cos\gamma\sin\delta)^2 + (a\cos\gamma\cos\delta)^2 = a^2$.

14. Prove $(\cos a\cos b - \sin a\sin b)^2$
$+ (\sin a\cos b + \cos a\sin b)^2 = 1$.

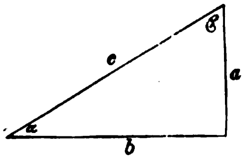
15. Of what angle is the secant double the sine?

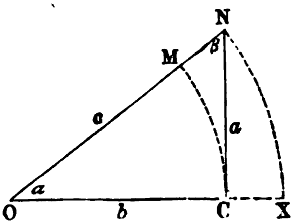
16. Of what angle is the secant four times the cosine?

CHAPTER III.

OF RIGHT TRIANGLES.

35. *Fundamental relations.* Let OCN be a right triangle of which a and b are the sides which contain the right angle, c the hypothenuse, α and β the angles opposite a and b respectively.





If we take ON as a radius and draw the arc NX from the centre O, the side NC will, by definition, represent the sine of XON, and OC its cosine, when the radius is ON. That is,

$$\frac{NC}{ON} = \sin\alpha; \qquad \frac{OC}{ON} = \cos\alpha. \tag{1}$$

We may show in the same way, by taking N as the centre and NO as the radius,

$$\frac{OC}{ON} = \sin\beta; \qquad \frac{NC}{ON} = \cos\beta. \tag{2}$$

We might also have deduced these equations from (1), because $\beta = 90° - \alpha$, whence

$$\sin\beta = \sin(90° - \alpha) = \cos\alpha;$$
$$\cos\beta = \cos(90° - \alpha) = \sin\alpha.$$

Again, by taking OC as a radius, we find

$$\frac{NC}{OC} = \tan\alpha = \cot\beta; \qquad \frac{OC}{NC} = \tan\beta = \cot\alpha. \tag{3}$$

Putting $NC = a$, $OC = b$, $ON = c$, the equations (1), (2), and (3) give the relations

$$\left.\begin{array}{l} a = c \sin\alpha \;\;\; = b \tan\alpha; \\ b = c \cos\alpha \;\;\; = a \cot\alpha; \\ c = a \operatorname{cosec}\alpha = b \sec\alpha. \end{array}\right\} \qquad (4)$$

We may express the same relations in terms of β, using the complementary functions, as follows:

$$\left.\begin{array}{l} a = c \cos\beta = b \cot\beta; \\ b = c \sin\beta = a \tan\beta; \\ c = a \sec\beta = b \operatorname{cosec}\beta. \end{array}\right\} \qquad (5)$$

These relations may be summed up in the following general theorems:

I. *The hypothenuse of any right triangle is equal to a side into the secant of its adjacent angle or the cosecant of its opposite angle.*

II. *A side is equal to the hypothenuse into the sine of the opposite angle or the cosine of the adjacent angle.*

III. *One side is equal to the other side into the tangent of the angle adjacent to that other side or the cotangent of the angle adjacent to itself.*

EXERCISE.

Show by the above equations how each side will be expressed in terms of the others when $\alpha = 30°$ and when $\alpha = 45°$, using the values of sin α, etc., already found—namely,

$$\sin 45° = \sqrt{\tfrac{1}{2}}; \qquad \tan 45° = 1; \text{ etc.,}$$

and $\quad \sin 30° = \tfrac{1}{2}; \qquad \cos 30° = \dfrac{\sqrt{3}}{2}; \qquad \tan 30° = \dfrac{1}{\sqrt{3}};$ etc.—

and show how the results agree with those of elementary geometry.

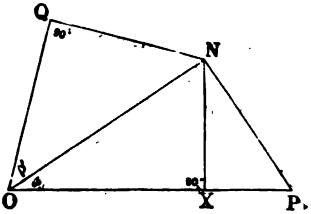
36. *Examples and exercises in expression.* In the accompanying figure OQN, ONP, and OXN are right angles, and we put

$$\alpha = \text{angle } XON;$$
$$\beta = \text{angle } NOQ.$$

It will also be noticed that

$$\text{Angle } XNP = \alpha$$

and Angle $ONX = XPN = 90° - \alpha.$



It is now required to express all the other lines in terms of OX and trigonometric functions of α and β.

Solution. We have

$$ON = OX \sec\alpha;$$
$$XN = OX \tan\alpha;$$
$$OP = ON \sec\alpha = OX \sec^2\alpha;$$
$$NP = OP \sin\alpha = OX \sin\alpha \sec^2\alpha = OX \tan\alpha \sec\alpha;$$

or $NP = ON \tan\alpha = OX \sec\alpha \tan\alpha$ (as before);

$$OQ = ON \cos\beta = OX \sec\alpha \cos\beta;$$
$$NQ = ON \sin\beta = OX \sec\alpha \sin\beta.$$

EXERCISES.

1. By the same process express OQ, QN, OX, NX, NP, and XP in terms of ON and trigonometric functions.

2. Express the same quantities in terms of NP.

3. Express NX separately in terms of OX and XP, and by multiplying the two values prove the geometric theorem that NX is a mean proportional between OX and XP.

4. In a right triangle the sides which contain the right angle are a and b, $(a > b)$, and δ is the difference of the angles at the base (hypothenuse). Express the length of the perpendicular from the vertex upon the base in two ways, and the lengths of the segments into which it divides the base each in one way. The expressions are all to be in terms of a, b and δ.

Ans. (*in part*). One expression for the perpendicular is

$$p = a \sin(45° - \delta).$$

Solution of Right Triangles.

Since in a right triangle one angle—the right angle—is given, only two other independent parts are required to solve the triangle. These two parts may be any two of the sides or one side and one angle. What parts soever are given, the remaining parts may be found by the equations (4) and (5). The following are all the essentially different cases.

37. CASE I. *Given the two sides a and b adjacent to the right angle.**

The first equation (4),

$$a = b \tan \alpha,$$

gives
$$\tan \alpha = \frac{a}{b}.$$

Therefore the quotient of the two sides gives the tangent of the angle opposite the dividend side.

From the tangent the angle α is itself found by the trigonometric tables; then sec α or cos α; then the hypothenuse c from the equation

$$c = b \sec \alpha = \frac{b}{\cos \alpha}.$$

Example. Given $a = 9$ metres, $b = 12$ metres, to find the remaining parts of the triangle.

Solution by numbers and measurement.

$$\tan \alpha = \tfrac{9}{12} = 0.75.$$

On the tangent line XN (§ 22) measure a distance from X equal to 0.75 of the radius OX; join the end of the distance to O, and measure the angle XON which the joining line makes with OX. This angle will be α. The length of the joining line divided by OX will be the secant, which multiplied by $b = 12$ will give the hypothenuse c.

The third angle, $\beta = 90° - \alpha$.

Logarithmic Solution.

log a	0.95 424	cos α	9.90 309
log b	1.07 918	log b	1.07 918
tan α	9.87 506	log c	1.17 609
α,	36° 52′.2	c,	15 metres.
β,	53° 7′.8		

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* It is recommended that in commencing this subject the student first solve a few of the problems by his own process, and without the use of any tables but those he may construct for himself by measurement as described in § 18.

EXERCISES.

1. Given $a = 4$, $b = 5$; find remaining parts.
2. " $a = 8$, $b = 5$; " "
3. " $a = 43.148$, $b = 84.107$; " "
4. " $a = 2.7938$, $b = 876.59$; " "
5. " $a = 759.28$, $b = 51.85$; " "
6. " $a = 8628$, $b = 27316$; " "

The first two exercises are made purposely simple, that they may be performed by measurement.

38. CASE II. *Given the hypothenuse and one side.*

Solution. From the equation

$$c \sin \alpha = a$$

we obtain

$$\sin \alpha = \frac{a}{c},$$

which may be used to find α when a and c are given. Then the remaining side is found by the equation

$$b = c \cos \alpha.$$

Example. Given $a = 13$, $c = 20$, to find the remaining parts.

Solution by numbers and measurement.

$$\sin \alpha = \frac{a}{c} = \frac{13}{20} = 0.65.$$

Take a distance equal to 0.65 of OX in the dividers, and find out what angle it will fit in the diagram (§ 15) to form a sine of an angle. This angle will be α. Measure off $\cos \alpha$ perpendicularly to OB, take its ratio to OX, and multiply it by $c = 5$. This will give the side b.

Logarithmic Solution.

log. 13	1.11 394	$\cos \alpha$	9.88 078
log. 20	1.30 103	log 20	1.30 103
$\sin \alpha$	9.81 291	$\log b$	1.18 181
α,	40° 32′.5	b,	15.199
β,	49° 27′.5		

EXERCISES.

1. Given $a = 7$, $c = 9$; find remaining parts.
2. " $b = 9$, $c = 16$; " " "
3. " $a = 82.143$, $c = 120.412$; " " "
4. " $b = 2.9235$, $c = 9.827$; " " "

5. A circle of radius r is drawn with its centre at a distance p from a straight line. What length will it cut out from the line? What will be the result if $r < p$?

39. CASE III. *Given an angle and one side, as α and a.*

Solution. The equations (a) give

$$b = \frac{a}{\tan\alpha} = a\cot\alpha = a\cot\beta;$$

$$c = \frac{a}{\sin\alpha} = a\operatorname{cosec}\alpha = a\sec\beta.$$

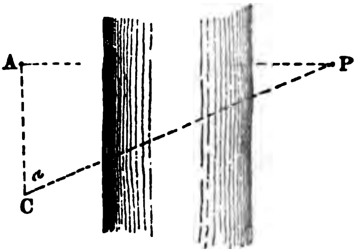
NOTE. One angle being given, the remaining one may be found from the equation

$$\beta = 90^\circ - \alpha.$$

EXERCISES.

1. Given $\alpha = 72^\circ\ 39'$, $c = 19.5$; find a and b.
2. " $\alpha = 16^\circ\ 25'.6$, $c = 10.925$; " "
3. " $\beta = 43^\circ\ 28'.4$, $b = 8.1273$; find a and c.
4. " $\beta = 8^\circ\ 29'.2$, $b = 0.9271$; " "

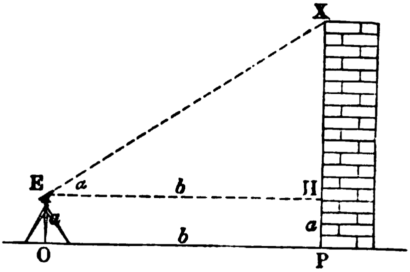
5. An engineer, desiring to find the distance from a point A on one bank of a river to a point P on the other bank, measured off a base line AC b yards in length, in a direction perpendicular to AP. He then measured the angle ACP, and found it to be α. What are the expressions for the distances AP and CP?



What will be the distance if $AC = 80$ yards and $\alpha = 85^\circ\ 22'.5$?

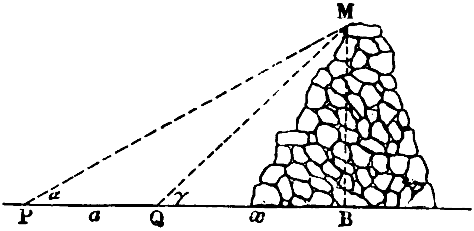
6. An engineer, desiring to determine the height of a vertical

wall, measured off a distance $PO = b$ feet on level ground, and then from a point E, a feet above the ground, measured the angle $HEX = \alpha$ between the line of sight EX and the horizontal line EH. Express the height PX of the wall algebraically in terms of a, b, and α, and compute the height when $b = 400$ feet, $a = 6$ feet, and the angle $\alpha = 22° 17'$.



Method of solution. Find the height HX and add $HP = a$.

7. Desiring to find the height of an inaccessible rock M above a plane, its angle BPM was measured from a point P and found to be α. The observer then advanced a metres towards the rock to Q, and there again measured the angle of elevation and found it to be γ. Express the height BM of the rock.



Method of solution. Let the vertical height $BM = h$ and $QB = x$. Then we have the two right triangles PBM and QBM, which give

$$h = (a + x) \tan \alpha;$$
$$h = x \tan \gamma.$$

From these two equations we obtain the following values of h and x:

$$x = \frac{a \tan \alpha}{\tan \gamma - \tan \alpha};$$

$$h = \frac{a \tan \alpha \tan \gamma}{\tan \gamma - \tan \alpha}.$$

Find the height when $a = 2000$ yards, $\alpha = 30° 28'$, and $\gamma = 40° 53'$.

8. The altitude of a triangle is 7.2648, and the angles at the base are 72° 29.3′ and 40° 30.5′ respectively. Compute the base

and sides. Also, find the general expression for the length of the base in terms of altitude and angles at base.

9. From the top of a tower 108 feet high the angles of depression of the top and bottom of another tower standing on the same horizontal plane are found to be 28° 56′ and 53° 41′ respectively. Find the distance between the towers, the height of the second tower, and the distance between the summits of the two towers.

10. In a circle of radius r, express the length of each side, and of the apothegm, of a regular inscribed polygon of n sides. Find first the special values for the triangle, square, pentagon, hexagon, and octagon. *Ans.* For the octagon: side $= 2r \sin 22\frac{1}{2}°$;
apothegm $= r \cos 22\frac{1}{2}°$.

11. If the side of a regular octagon is 10 metres, what are the radii of the inscribed and circumscribed circles?

12. At what altitude is the sun when a tower 20 metres high casts a shadow 75 metres long upon a horizontal plane?

13. It was found that the length of the shadow of a monument upon a horizontal plane diminished 22 metres when the sun's altitude increased from 30° to 45°. What was the height of the monument?

14. If β is one of the acute angles of a right triangle, and c its hypothenuse, express the altitude in terms of c and β.

15. Two lighthouses, each 30 metres above the sea and 500 metres apart, are seen by a ship in line with them to differ 1° in elevation above the horizon. What is the distance from the nearer, supposing the ocean a plane?

16. The great pyramid of Gizeh is 762 feet square at its base, and each side makes an angle of 51° 51′ with the horizon. Find—

(*a*) Its height if continued to its apex.

(*b*) The slope of its edges.

Actually, instead of being continued to its apex, it terminates in a platform 32 feet square. Find—

(*c*) The perpendicular height of this platform above the base.

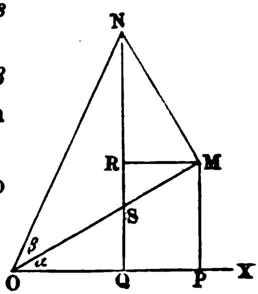
(*d*) The length of each edge from the corner of the platform to the corner of the base.

CHAPTER IV.

RELATIONS BETWEEN FUNCTIONS OF SEVERAL ANGLES.

The Addition and Subtraction Theorems.

40. Problem. *To express the sine and cosine of the sum of two angles in terms of the sines and cosines of the angles themselves.*



Solution. Let $XOM = \alpha$ and $MON = \beta$ be the two angles. $XON = \alpha + \beta$ is then their sum.

Let ON be the unit radius. From N drop

$$NM \perp OM; \qquad NQ \perp OX.$$

From M drop

$$MP \perp OX; \qquad MR \perp NQ.$$

Then
$$\left.\begin{aligned}\sin(\alpha+\beta) &= NQ = NR + MP;\\ \cos(\alpha+\beta) &= OQ = OP - MR.\end{aligned}\right\} \qquad (1)$$

OQS and NMS being right angles, we have

Angle $RNM =$ comp. $RSM =$ comp. $OSQ = SOQ = \alpha$.

By § 35,

$$\begin{aligned}NM &= \sin\beta;\\ NR &= NM\cos\alpha = \sin\beta\cos\alpha;\\ OM &= \cos\beta;\\ MP &= OM\sin\alpha = \cos\beta\sin\alpha;\end{aligned}$$

whence, from (1),

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \qquad (2)$$

We find in the same way

$$\begin{aligned}OP &= OM\cos\alpha = \cos\beta\cos\alpha;\\ RM &= NM\sin\alpha = \sin\beta\sin\alpha;\end{aligned}$$

whence, from (1),

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \qquad (3)$$

The formulæ (2) and (3) constitute the *addition theorem* of trigonometry.

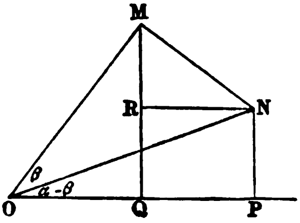
To find the corresponding subtraction theorem it is only necessary to change the sign of β. We have

$$\sin(-\beta) = -\sin\beta;$$
$$\cos(-\beta) = \cos\beta.$$

Therefore, changing β to $-\beta$ in (2) and (3), we find

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta; \qquad (4)$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \qquad (5)$$

It is, however, interesting to show how these equations may be obtained independently by a geometrical construction.



Let POM be the angle α, and NOM the angle β. Then

$$PON = \alpha - \beta.$$

Take ON for the unit radius and drop

$$NP \perp OP; \qquad NM \perp OM;$$
$$MQ \perp OP; \qquad NR \perp MQ.$$

Then $\sin(\alpha-\beta) = PN = QR = MQ - MR;$

$$\cos(\alpha-\beta) = OP = OQ + RN.$$

Because OMN is a right angle,

$$NMR = \text{comp. } OMQ = MOQ = \alpha;$$
$$MN = \sin\beta;$$
$$OM = \cos\beta;$$
$$MR = MN\cos NMR = \sin\beta\cos\alpha;$$
$$MQ = OM\sin MOQ = \cos\beta\sin\alpha;$$
$$OQ = OM\cos MOQ = \cos\beta\cos\alpha;$$
$$RN = MN\sin NMR = \sin\beta\sin\alpha.$$

Making these substitutions, we have the results (4) and (5) for $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$.

41. *Sine and cosine of twice an angle.* If we suppose $\beta = \alpha$, we have from (2) and (3) expressions for the sine and cosine of the double of an angle, namely:

$$\sin 2\alpha = \sin\alpha\cos\alpha + \sin\alpha\cos\alpha, \qquad (6)$$

or

$$\sin 2\alpha = 2\sin\alpha\cos\alpha;$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha). \qquad (7)$$

Also, by putting for $\cos^2 \alpha$ its value, $1 - \sin^2 \alpha$, and *vice versa*, we have

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1. \tag{7'}$$

EXERCISES.

1. Because $90° = 60° + 30°$, we have, by putting $\alpha = 60°$ and $\beta = 30°$ in the equation (2),

$$\sin 90° = \sin 60° \cos 30° + \cos 60° \sin 30° = 1.$$

It is required to test this equation by substituting the numerical values of the sines and cosines of 30°, 60°, and 90° (§30), and to test in the same way the equations obtained by putting $\alpha = 60°$ and $\beta = 30°$ in (3), (4) and (5).

2. Because $\alpha = \alpha - x + x$, we have, from (2),

$$\sin \alpha = \sin (\alpha - x) \cos x + \cos (\alpha - x) \sin x.$$

It is required to write the corresponding equations obtained by making the same substitution in (3), and the equations obtained in the same way from (4) and (5) by the identical equation

$$\alpha = \alpha + x - x.$$

3. Derive the addition theorem for the cosine from that for the sine by substituting $\gamma - 90°$ or $90° - \gamma$ for β in the equation (2), and applying the equations (18) of § 29 (Chap. II.).

4. By means of the addition theorem prove the equations

$$\sin (\alpha + \beta + \gamma) = \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma - \sin \alpha \sin \beta \sin \gamma;$$

$$\cos (\alpha + \beta + \gamma) = \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \beta \cos \gamma - \sin \alpha \cos \beta \sin \gamma - \cos \alpha \sin \beta \sin \gamma.$$

NOTE. This is readily done by putting $\alpha + \beta$ for α, and γ for β, in the equations (2) and (3), thus giving

$$\sin (\alpha + \beta + \gamma) = \sin (\alpha + \beta) \cos \gamma + \cos (\alpha + \beta) \sin \gamma;$$

$$\cos (\alpha + \beta + \gamma) = \cos (\alpha + \beta) \cos \gamma - \sin (\alpha + \beta) \sin \gamma;$$

and then developing by the addition theorem.

5. Prove the following values of $\sin 3\alpha$ and $\cos 3\alpha$:

$$\sin 3\alpha = 3 \sin \alpha \cos^2 \alpha - \sin^3 \alpha;$$

$$\cos 3\alpha = \cos^3 \alpha - 3 \sin^2 \alpha \cos \alpha.$$

6. Transform the expression

$$a \cos (\alpha + x) + b \cos (\beta + x) + c \cos (\gamma + x)$$

into

$$\cos x\,(a\cos\alpha + b\cos\beta + c\cos\gamma)$$
$$- \sin x\,(a\sin\alpha + b\sin\beta + c\sin\gamma).$$

7. If we have

$$a\cos\alpha + b\cos\beta + c\cos\gamma = 0$$

and

$$a\sin\alpha + b\sin\beta + c\sin\gamma = 0,$$

prove that we must also have

$$a\sin(\alpha + x) + b\sin(\beta + x) + c\sin(\gamma + x) = 0,$$

whatever be the value of x.

42. *The Addition Theorem for tangents.* Dividing equation (2) by (3), we have

$$\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \tan(\alpha+\beta) = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Dividing both numerator and denominator of the last member by $\cos\alpha\cos\beta$, the equation becomes

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}. \qquad (8)$$

We obtain in the same way from (4) and (5),

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}. \qquad (9)$$

Putting $\beta = \alpha$ in (8) we have

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}. \qquad (10)$$

EXERCISES.

1. Assuming $\beta = 180°$, prove by (8) of this section that

$$\tan(\alpha + 180°) = \tan\alpha.$$

2. Assuming $\alpha = 30°$, substitute in (10) the value of $\tan 30°$ given in § 30, and thus obtain the value of $\tan 60°$.

43. *Products of sines and cosines.* Taking the sum and difference of equations (2) and (4) of § 40, and reversing the members of the equation, we find

$$\left.\begin{aligned} 2\sin\alpha\cos\beta &= \sin(\alpha+\beta) + \sin(\alpha-\beta);\\ 2\cos\alpha\sin\beta &= \sin(\alpha+\beta) - \sin(\alpha-\beta). \end{aligned}\right\} \qquad (11)$$

In the same way we find from (3) and (5),

$$\left.\begin{aligned} 2\cos\alpha\cos\beta &= \cos(\alpha+\beta)+\cos(\alpha-\beta); \\ 2\sin\alpha\sin\beta &= -\cos(\alpha+\beta)+\cos(\alpha-\beta). \end{aligned}\right\} \qquad (12)$$

EXERCISES.

1. Prove that if $\alpha - \beta = 90°$,

then $\cos(\alpha+\beta) = 2\cos\alpha\cos\beta = \sin 2\alpha = -\sin 2\beta.$

2. Prove that if $\alpha + \beta = 180°$,

then $\sin(\alpha-\beta) = 2\sin\alpha\cos\beta = \sin 2\beta = -\sin 2\alpha.$

44. *Sum of sines and cosines.* If in the four equations (11) and (12) we put

$$\alpha + \beta = x$$

and

$$\alpha - \beta = y,$$

we shall have

$$\alpha = \tfrac{1}{2}(x+y)$$

and

$$\beta = \tfrac{1}{2}(x-y).$$

By substituting these values in (11) and (12) and reversing the members of the equation, we find

$$\left.\begin{aligned} \sin x + \sin y &= 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y); \\ \sin x - \sin y &= 2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y); \\ \cos x + \cos y &= 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y); \\ -\cos x + \cos y &= 2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y). \end{aligned}\right\} \qquad (13)$$

Dividing the first of this group of equations by the third, we get

$$\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\tfrac{1}{2}(x+y). \qquad (14)$$

Dividing the second by the fourth, we get

$$\frac{\sin x - \sin y}{\cos y - \cos x} = \cot\tfrac{1}{2}(x+y). \qquad (15)$$

If in the last two equations of (13) we suppose $y = 0$, which makes $\cos y = 1$, they become

$$\left.\begin{aligned} 1 + \cos x &= 2\cos^2\tfrac{1}{2}x; \\ 1 - \cos x &= 2\sin^2\tfrac{1}{2}x; \end{aligned}\right\} \qquad (16)$$

a pair of equations which frequently come into use.

If we put $\alpha = \tfrac{1}{2}x$, these equations become

$$\left.\begin{aligned} 1 + \cos 2\alpha &= 2\cos^2\alpha; \\ 1 - \cos 2\alpha &= 2\sin^2\alpha; \end{aligned}\right\} \qquad (16')$$

which may also be derived directly from (7').

Dividing the second of (16) and (16′) by the first, we find

$$\frac{1-\cos x}{1+\cos x} = \tan^2 \tfrac{1}{2}x; \qquad (17)$$

$$\frac{1-\cos 2\alpha}{1+\cos 2\alpha} = \tan^2 \alpha. \qquad (17')$$

45. *The problem of dimidiation.* It is often required to express the sine or cosine of half an angle in terms of the sine or cosine of the entire angle. To effect this let us put in equations (6) and (7) $2\alpha = \gamma$. They then become

$$\left.\begin{aligned} \sin\gamma &= 2\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\gamma;\\ \cos\gamma &= \cos^2\tfrac{1}{2}\gamma - \sin^2\tfrac{1}{2}\gamma,\\ &= 1 - 2\sin^2\tfrac{1}{2}\gamma,\\ &= 2\cos^2\tfrac{1}{2}\gamma - 1. \end{aligned}\right\} \qquad (18)$$

Let our first problem be:

Given, $\cos\gamma$;
Required, $\sin\frac{1}{2}\gamma$.

Solving the equation $\cos\gamma = 1 - 2\sin^2\frac{1}{2}\gamma$, we obtain

$$\sin\tfrac{1}{2}\gamma = \sqrt{\frac{1-\cos\gamma}{2}}. \qquad (19)$$

Let our next problem be:

Given, $\cos\gamma$;
Required, $\cos\frac{1}{2}\gamma$.

Solving the equation $\cos\gamma = 2\cos^2\frac{1}{2}\gamma - 1$, we obtain

$$\cos\tfrac{1}{2}\gamma = \sqrt{\frac{1+\cos\gamma}{2}}. \qquad (20)$$

Let our third problem be:

Given, $\sin\gamma$;
Required, $\sin\frac{1}{2}\gamma$.

In the equation

$$\sin\gamma = 2\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\gamma$$

we put $\cos\frac{1}{2}\gamma = \sqrt{1-\sin^2\frac{1}{2}\gamma}$ and square both members, obtaining

$$\sin^2\gamma = 4\sin^2\tfrac{1}{2}\gamma\,(1-\sin^2\tfrac{1}{2}\gamma).$$

Reducing,

$$\sin^4\tfrac{1}{2}\gamma - \sin^2\tfrac{1}{2}\gamma + \frac{\sin^2\gamma}{4} = 0.$$

Considering $\sin\frac{1}{2}\gamma$ as an unknown quantity, this equation can be solved as a quadratic. Transposing the last term and adding $\frac{1}{4}$ to each member, it becomes

$$\sin^4 \tfrac{1}{2}\gamma - \sin^2 \tfrac{1}{2}\gamma + \tfrac{1}{4} = \frac{1 - \sin^2 \gamma}{4} = \frac{\cos^2 \gamma}{4};$$

extracting the square root of both members,

$$\sin^2 \tfrac{1}{2}\gamma - \tfrac{1}{2} = \pm \frac{\cos \gamma}{2};$$

whence, by solving,

$$\sin \tfrac{1}{2}\gamma = \sqrt{\frac{1 \pm \cos \gamma}{2}}. \qquad (21)$$

Let our fourth problem be:

Given, sin γ;
Required, cos $\frac{1}{2}\gamma$.

In the equation

$$2 \sin \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\gamma = \sin \gamma$$

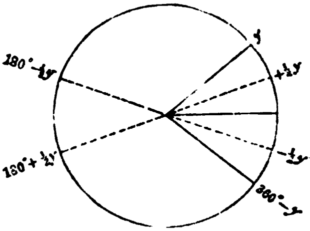
we put $\sin \frac{1}{2}\gamma = \sqrt{1 - \cos^2 \frac{1}{2}\gamma}$, and then proceed as before.

Solving with respect to $\cos \frac{1}{2}\gamma$, we shall find

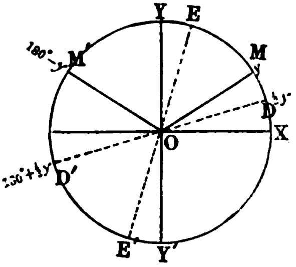
$$\cos \tfrac{1}{2}\gamma = \frac{\sqrt{1 \mp \cos \gamma}}{2}. \qquad (22)$$

We have now to study the different values which these expressions for $\sin \frac{1}{2}\gamma$ and $\cos \frac{1}{2}\gamma$ may have in consequence of the double signs of the surds and the double signs under the radical sign.

Equations (19) and (20) show that if the cosine of an angle γ is given, the sine and cosine of half that angle may have either of two opposite values. The geometrical explanation of this is that to $\cos \gamma$ may correspond either of two angles, γ or $360° - \gamma$. The halves of the general measure of γ are $\frac{1}{2}\gamma$ and $\frac{1}{2}\gamma + 180°$ (§ 10). The halves of the general measure of $360° - \gamma$ are $180° - \frac{1}{2}\gamma$ and $-\frac{1}{2}\gamma$ (see diagram).



Therefore the half angle may have either of four values. But because $\sin(180° + \frac{1}{2}\gamma) = \sin(-\frac{1}{2}\gamma)$, and $\sin(180° - \frac{1}{2}\gamma) = \sin \frac{1}{2}\gamma$, the sines and cosines of these four angles will have only two different values. This result agrees with equations (19) and (20).



But suppose the *sine* of γ to be given. The angle γ may then have either of two supplementary values, γ and $180° - \gamma$. The halves of the general measure of these angles will be $\frac{1}{2}\gamma$, $90° - \frac{1}{2}\gamma$, $180° + \frac{1}{2}\gamma$, and $270° + \frac{1}{2}\gamma$. The sines and cosines of these four angles are all different. Therefore the algebraic expression for the sines ought to be susceptible of four values instead of two, as in the first case. Now this is true of equations (5) and (6), because they show that

$$\sin \tfrac{1}{2}\gamma = +\sqrt{\frac{1+\cos\gamma}{2}} \text{ or } -\sqrt{\frac{1+\cos\gamma}{2}}$$

$$\text{or } +\sqrt{\frac{1-\cos\gamma}{2}} \text{ or } -\sqrt{\frac{1-\cos\gamma}{2}},$$

and
$$\cos \tfrac{1}{2}\gamma = \sqrt{\frac{1-\cos\gamma}{2}} \text{ or } -\sqrt{\frac{1-\cos\gamma}{2}}$$

$$\text{or } +\sqrt{\frac{1+\cos\gamma}{2}} \text{ or } -\sqrt{\frac{1+\cos\gamma}{2}}.$$

EXERCISES.

1. We have already found $\sin 30° = \frac{1}{2}$; from this find the sine, cosine, tangent, and cotangent of 15° and of 75°.

2. From the values of the six trigonometric functions for 45° (§ 30) find those for 22½°.

3. From the values for 18° find those for 36°.

46. *Miscellaneous relations.* The following equations are of occasional use in the applications of trigonometry, and can all be derived from the formulæ of the last two chapters. Their derivations are therefore presented as an interesting exercise.

1. $\sin(45° - x) = \cos(45° + x)$.

2. $\sin x = \dfrac{\cos x}{\cot x}$.

3. $\sin x = \sin(60° + x) - \sin(60° - x)$.

4. $\sin x = \dfrac{2 \tan \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x}.$

5. $\sin x = \dfrac{2}{\tan \frac{1}{2}x + \cot \frac{1}{2}x}.$

6. $\sin x = \dfrac{\sin (30^\circ + x) - \sin (30^\circ - x)}{\sqrt{3}}.$

7. $\sin x = 2 \sin^2 (45^\circ + \frac{1}{2}x) - 1 = 1 - 2 \sin^2 (45^\circ - \frac{1}{2}x).$

8. $\cos \varepsilon = \sin \varepsilon \cot \varepsilon.$

9. $\cos \varepsilon = \dfrac{1 - \tan^2 \frac{1}{2}\varepsilon}{1 + \tan^2 \frac{1}{2}\varepsilon}.$

10. $\cos a = \dfrac{\cot \frac{1}{2}a - \tan \frac{1}{2}a}{\cot \frac{1}{2}a + \tan \frac{1}{2}a}.$

11. $\cos a = \dfrac{1}{1 + \tan a \tan \frac{1}{2}a}.$

12. $\cos a = 2 \cos (45^\circ + \frac{1}{2}a) \cos (45^\circ - \frac{1}{2}a).$

13. $\cos a = \dfrac{2}{\tan (45^\circ + \frac{1}{2}a) + \tan (45^\circ - \frac{1}{2}a)}.$

14. $\tan \theta = \dfrac{2 \cot \frac{1}{2}\theta}{\cot^2 \frac{1}{2}\theta - 1}.$

15. $\tan \theta = \dfrac{2}{\cot \frac{1}{2}\theta - \tan \frac{1}{2}\theta}.$

16. $2 \cot 2a = \cot a - \tan a.$

17. $\tan \theta = \dfrac{\sin 2\theta}{1 + \cos 2\theta}.$

18. $\tan a = \dfrac{\tan (45^\circ + \frac{1}{2}a) - \tan (45^\circ - \frac{1}{2}a)}{2}.$

19. $\dfrac{\sin (A + B)}{\sin (A - B)} = \dfrac{\tan A + \tan B}{\tan A - \tan B} = \dfrac{\cot B + \cot A}{\cot B - \cot A}.$

20. $\dfrac{\cos (A + B)}{\cos (A - B)} = \dfrac{\cot B - \tan A}{\cot B + \tan A} = \dfrac{\cot A - \tan B}{\cot A + \tan B}.$

21. $\cos a \cos b = \cos^2 \frac{1}{2}(a + b) - \sin^2 \frac{1}{2}(a - b).$

22. $\sin a \sin b = \cos^2 \frac{1}{2}(a - b) - \cos^2 \frac{1}{2}(a + b).$

23. $\tan a + \cot a = 2 \operatorname{cosec} 2a.$

24. $\sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta = \sin (\alpha + \beta) \sin (\alpha - \beta).$

25. $\cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta = \cos (\alpha + \beta) \cos (\alpha - \beta).$

CHAPTER V.

TRIGONOMETRIC PROBLEMS.

47. Problem I. *Having given two equations of the form*

$$r \sin \varphi = a,$$
$$r \cos \varphi = b,$$

where a and b are given numbers, it is required to compute r and φ.

Solution. Dividing the first equation by the second, we find

$$\frac{\sin \varphi}{\cos \varphi} = \tan \varphi = \frac{a}{b}.$$

From this value of $\tan \varphi$ we find φ itself, then the sine or cosine of φ, and then r from either of the equations

$$r = \frac{a}{\sin \varphi} = \frac{b}{\cos \varphi}.$$

Example. If $r \sin \varphi = 332.76$, and $r \cos \varphi = 290.08$, it is required to find r and φ.

log sin φ, 9.877 25 (5) log cos φ, 9.817 64 (6)
" r sin φ, 2.522 13 (1) " tan φ, 0.059 61 (3)
" r cos φ, 2.462 52 (2) φ, 48° 55′.2 (4)
" r = log r sin φ − log sin φ, 2.644 88 (7)
" r = log r cos φ − log cos φ, 2.644 88 (8)
r, 441.45 (9)

The numbers in brackets show the order in which the numbers of the computation are written. In writing log $r \sin \varphi$ and $r \cos \varphi$ spaces are left for inserting log sin φ and log cos φ after φ is found, so that either of the latter may be subtracted to obtain r. It is generally best to obtain r from both $r \sin \varphi$ and $r \cos \varphi$, because then if any mistake is made in φ it will be shown by a difference in the results.

On the other hand, a practical computer will not write down either $\sin \varphi$ or $\cos \varphi$, but will subtract in his head and write down $\log r$ only.

EXERCISES.

1. Given $r \sin \varphi = 1.297\,43$, $r \cos \varphi = 6.002\,4$; find r and φ.
2. " " $= 0.082\,19$, " $= 0.128\,8$; " "
3. " " $= 194\,683$, " $= 8460.7$; " "

48. *Distinction of quadrants.* In the preceding examples we have supposed $r \sin \varphi$ and $r \cos \varphi$ to be positive, and have taken φ in the first quadrant. But either or both of these quantities may be negative. Whatever their signs, there are always two values of φ, differing by 180°, corresponding to any given value of $\tan \varphi$ (§ 31). Hence the problem admits of two solutions in all cases. In the one r will be positive, in the other negative.

But in practice only that solution is sought which gives a positive value of r. This being the case, $\sin \varphi$ and $\cos \varphi$ must have the same algebraic signs as the given quantities $r \sin \varphi$ and $r \cos \varphi$ respectively. Now consider each case in order:

I. $r \sin \varphi$ and $r \cos \varphi$ *both positive.* The angle φ must then be taken in the first quadrant, because only in this quadrant are $\sin \varphi$ and $\cos \varphi$ *both* positive.

II. $r \sin \varphi$ *positive* and $r \cos \varphi$ *negative.* Sin φ is positive only in quadrants (1) and (2) (§ 21), and $\cos \varphi$ is negative only in quadrants (2) and (3). Hence the requirement of signs can be fulfilled only in the second quadrant, and

$$90^\circ < \varphi < 180^\circ.$$

III. $r \sin \varphi$ and $r \cos \varphi$ *both negative.* The only quadrant in which sine and cosine are both negative is the third. Hence in this case

$$180^\circ < \varphi < 270^\circ.$$

IV. $r \sin \varphi$ *negative* and $r \cos \varphi$ *positive.* The only quadrant in which $\sin \varphi$ is negative and $\cos \varphi$ positive is the fourth. Hence in this case

$$270^\circ < \varphi < 360^\circ.$$

EXERCISES.

1. Given $r \sin \varphi = -237.09$, $r \cos \varphi = +192.91$; find r and φ.
2. " " $+2713.8$, " -9269.2; " "
3. " " -1.9634, " -0.09654; " "
4. " " -3.6925, " $+396.72$; " "
5. " " -18.005, " -2.6943; " "
6. Given
$$r \sin (\varphi + 47^\circ\ 50') = 7.2693,$$
$$r \cos (\varphi + 47^\circ\ 50') = -12.2916,$$
to find r and φ.

NOTE. In this last exercise compute the value of $(\varphi + 47^\circ\ 50')$ as if it were one quantity, and subtract the angle $47^\circ\ 50'$ from the result.

7. Given
$$r \sin (\theta + x) = -249.88,$$
$$r \cos (\theta + x) = -92.62,$$
$$u \sin (\theta - x) = 702.02,$$
$$u \cos (\theta - x) = 516.93;$$
find the values of r, u, θ, and x.
8. Given
$$r (\sin \theta + \cos \theta) = 298.07,$$
$$r (\sin \theta - \cos \theta) = 96.04;$$
find r and θ.

49. PROBLEM II. *Having given two equations of the form*
$$x \cos \alpha + y \sin \alpha = p,$$
$$x \sin \alpha - y \cos \alpha = q,$$
it is required to find the values of x and y.

The elimination is conducted by the method of addition and subtraction, as follows:

Multiply the first equation by $\cos \alpha$ and the second by $\sin \alpha$. We thus have
$$x \cos^2 \alpha + y \sin \alpha \cos \alpha = p \cos \alpha;$$
$$x \sin^2 \alpha - y \sin \alpha \cos \alpha = q \sin \alpha.$$
Now adding these equations together and remembering that
$$\sin^2 \alpha + \cos^2 \alpha = 1,$$
we have
$$x = p \cos \alpha + q \sin \alpha,$$
which is the required value of x.

Next multiply the first equation by sin α and the second by $-\cos\alpha$. We have

$$x\sin\alpha\cos\alpha + y\sin^2\alpha = p\sin\alpha;$$
$$-x\sin\alpha\cos\alpha + y\cos^2\alpha = -q\cos\alpha.$$

By addition,

$$y = p\sin\alpha - q\cos\alpha.$$

It will be noticed that in these equations x and y are given in terms of p and q by equations of the same form as the original ones.

EXERCISE.

Find the value of p and q from the equations

$$p\sin\alpha + q\cos\alpha = a;$$
$$p\cos\alpha - q\sin\alpha = b.$$

50. PROBLEM III. *From the equations*

$$r\sin(\beta + \varepsilon) = a,$$
$$r\sin(\beta + \theta) = b,$$

to find the values of r *and* β,*—the other four quantities,* a, b, ε, *and* θ, *being supposed known.*

Solution. Developing the sines of $\beta + a$ and $\beta + \theta$ (§ 40), we have

$$\left.\begin{aligned} r\sin\beta\cos\varepsilon + r\cos\beta\sin\varepsilon = a; \\ r\sin\beta\cos\theta + r\cos\beta\sin\theta = b. \end{aligned}\right\} \qquad (a)$$

Regarding $r\sin\beta$ and $r\cos\beta$ as the two unknown quantities, we see that their coefficients are $\cos\varepsilon$, $\sin\varepsilon$, $\cos\theta$ and $\sin\theta$. Multiplying the first equation by $\cos\theta$ and the second by $\cos\varepsilon$, we have

$$r\sin\beta\cos\varepsilon\cos\theta + r\cos\beta\sin\varepsilon\cos\theta = a\cos\theta;$$
$$r\sin\beta\cos\varepsilon\cos\theta + r\cos\beta\cos\varepsilon\sin\theta = b\cos\varepsilon.$$

Subtracting,

$$r\cos\beta(\sin\varepsilon\cos\theta - \cos\varepsilon\sin\theta) = a\cos\theta - b\cos\varepsilon.$$

Noting that the coefficient of $r\cos\beta$ is $\sin(\varepsilon - \theta)$, we find, by division,

$$r\cos\beta = \frac{a\cos\theta - b\cos\varepsilon}{\sin(\varepsilon - \theta)}.$$

To find the value of $r\sin\beta$ we multiply the first equation (a) by $\sin\theta$ and the second by $\sin\varepsilon$, and subtract. We thus find

$$r\sin\beta(\sin\varepsilon\cos\theta - \cos\varepsilon\sin\theta) = b\sin\varepsilon - a\sin\theta.$$

Hence $$r \sin \beta = \frac{b \sin \varepsilon - a \sin \theta}{\sin (\varepsilon - \theta)}.$$

Supposing the numerical values of a, b, ε, and θ to be given, these equations give the values of $r \sin \beta$ and $r \cos \beta$, from which r and β can be computed by Problem I.

EXERCISES.

Find the values of $r \sin \varphi$ and $r \cos \varphi$ from the following equations by the preceding method:

1. $r \cos (\varphi + \varepsilon) = a$, $r \cos (\varphi - \theta) = b$.

Ans. $$r \sin \varphi = \frac{b \cos \varepsilon - a \cos \theta}{\sin (\varepsilon + \theta)};$$
$$r \cos \varphi = \frac{b \sin \varepsilon + a \sin \theta}{\sin (\varepsilon + \theta)}.$$

2. $r \cos (\varphi + \varepsilon) = a$, $r \sin (\varphi + \theta) = b$.

Ans. $$r \sin \varphi = \frac{b \cos \varepsilon - a \sin \theta}{\cos (\varepsilon - \theta)};$$
$$r \cos \varphi = \frac{b \sin \varepsilon + a \cos \theta}{\cos (\varepsilon - \theta)}.$$

3. $r \sin (\varphi + \varepsilon) = a \cos \gamma$, $r \cos (\varphi - \varepsilon) = a \sin \gamma$.

Ans. $$r \sin \varphi = \frac{a \cos (\gamma + \varepsilon)}{\cos 2\varepsilon};$$
$$r \cos \varphi = \frac{a \sin (\gamma - \varepsilon)}{\cos 2\varepsilon}.$$

4. Find the values of r and φ from the equations
$$r \cos (\varphi + \theta) = 3.790\,8,$$
$$r \cos (\varphi - \theta) = 2.060\,7,$$
when $\theta = 31° 27'.4$.

51. PROBLEM IV. *To reduce an expression of the form*
$$a \sin \theta + b \cos \theta$$
to a monomial, a and b being given quantities.

Solution. Determine the values of two auxiliary quantities k and ε from the equations
$$k \cos \varepsilon = a,$$
$$k \sin \varepsilon = b,$$
as in Problem I.

The given expression will then become, by substitution,

$$k\cos\varepsilon\sin\theta + k\sin\varepsilon\cos\theta = k\sin(\theta+\varepsilon).$$

We might equally have supposed $k\sin\varepsilon = a$ and $k\cos\varepsilon = b$, when the given expression would have become $k\cos(\theta-\varepsilon)$.

Example. Reduce the expression

$$1239.3\sin x - 724.6\cos x$$

to a monomial.

$$\begin{array}{lrrr} k\sin\varepsilon = & -\ 724.6 & \log = & -\ 2.860\,10 \\ k\cos\varepsilon = & 1239.3 & \log = & 3.093\,18 \\ \cline{4-4} & & \log\tan\varepsilon, & -\ 9.766\,92 \\ & & \varepsilon, & -\ 30^\circ\ 18'.8 \\ & & \log\cos\varepsilon, & 9.936\,15 \\ & & \log k, & 3.157\,03 \\ & & k, & 1435.6 \end{array}$$

We therefore have

$$1239.3\sin x - 724.6\cos x = 1435.6\sin(x - 30^\circ\ 18'.8).$$

EXERCISES.

Reduce to monomials the expressions:

1. $27.615\cos\mu - 23.208\sin\mu$.
2. $3.600\,3\sin(\theta - 7^\circ\ 52'.6) + 5.907\,0\sin(\theta + 53^\circ\ 57'.6)$.
3. Reduce to a monomial the expression

$$92.65\sin\theta\cos\alpha - 196.23\cos\theta\sin\alpha,$$

when $\alpha = 162^\circ\ 48'.7$.

4. Reduce to a monomial the expression

$$\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos H,$$

when $\alpha = 62^\circ\ 39'.5$ and $H = 22^\circ\ 36'.8$.

52. PROBLEM V. *To reduce an expression of the form*

$$a\sin x\cos\theta + b\cos x\sin\theta \qquad (1)$$

to a form which shall not contain the product of any two trigonometric functions.

Solution. We have, from § 43,

$$\sin x\cos\theta = \tfrac{1}{2}\sin(x+\theta) + \tfrac{1}{2}\sin(x-\theta);$$
$$\cos x\sin\theta = \tfrac{1}{2}\sin(x+\theta) - \tfrac{1}{2}\sin(x-\theta).$$

Making these substitutions in the expression (1), it becomes

$$\tfrac{1}{2}(a+b)\sin(x+\theta)+\tfrac{1}{2}(a-b)\sin(x-\theta)$$

EXERCISES.

Clear from products of sines and cosines:

1. $m\cos\alpha\sin\beta - n\sin\alpha\cos\beta$.
2. $a\cos\theta\cos\mu + b\sin\theta\sin\mu$.
3. $a\cos\theta\cos\mu - b\sin\theta\sin\mu$.

53. PROBLEM VI. From the equations

$$\left.\begin{aligned} r\cos\beta\cos\lambda &= a,\\ r\cos\beta\sin\lambda &= b,\\ r\sin\beta &= c, \end{aligned}\right\} \qquad (1)$$

to find the values of r, β, and λ, the values of a, b, and c being given.

Method of solution. Dividing the second equation by the first we obtain

$$\tan\lambda = \frac{b}{a}; \qquad (2)$$

from this equation we find λ, and then $\sin\lambda$ or $\cos\lambda$ from the tables. Then

$$r\cos\beta = \frac{a}{\cos\lambda} = \frac{b}{\sin\lambda} \qquad (3)$$

can be computed. The value of $r\sin\beta$ being given by the third equation (1), the values of r and β are found by Problem I.

Example. Find r, β, and λ from the equations

$$\begin{aligned} r\cos\beta\cos\lambda &= -\ 53.953;\\ r\cos\beta\sin\lambda &= +\ 197.207;\\ r\sin\beta &= -\ 39.062. \end{aligned}$$

Work

$\log r\cos\beta\cos\lambda$,	$-\,1.732\,01$
$\log r\cos\beta\sin\lambda$,	$2.294\,93$
$\sin\lambda$,	$9.984\,33$
$\log\tan\lambda$,	$-\,0.562\,92$
λ,	105° 18′.0
$\log r\cos\beta$,	$2.310\,60$

$\log\cos\beta$,	$9.992\,21$
$\log r\cos\beta$,	$2.310\,60$
$\log r\sin\beta$,	$-\,1.591\,75$
$\log\tan\beta$,	$-\,9.281\,15$
$\log r$,	$2.318\,39$
r,	208.16
β,	− 10° 49′

EXERCISES.

Find the values of r, β, and λ from:

$$
\begin{aligned}
&1. \quad r\cos\beta\cos\lambda = 1.271\,83; \\
& \quad r\cos\beta\sin\lambda = -\,0.981\,52; \\
& \quad r\sin\beta = 0.890\,02. \\
&2. \quad r\sin\beta\sin\lambda = 19.765\,3; \\
& \quad r\sin\beta\cos\lambda = -\,7.192\,8; \\
& \quad r\cos\beta = 12.124\,2.
\end{aligned}
$$

MISCELLANEOUS EXERCISES.

1. Compute ω and r from the equations

$$1.268\,22\sin\omega = 0.948\,30 + r\sin(25°\,27'.2);$$
$$1.268\,22\cos\omega = 0.281\,16 + r\cos(25°\,27'.2).$$

First eliminate r.

Ans. $\omega = 60°\,53'.8;$
$r = 0.371\,7.$

2. Find ω and x from the equations

$$3\sin\omega + \cos\omega = 2x. \qquad \text{Ans. } \omega = 71°\,34';$$
$$\sin\omega + 2\cos\omega = x. \qquad x = \sqrt{\tfrac{5}{2}}.$$

After finding $\sin\omega$ and $\cos\omega$ in terms of x employ the equation

$$\sin^2\omega + \cos^2\omega = 1.$$

Find x from the following quadratic equations, and express the results without surds:

3. $x^2 + 1 = \dfrac{2x}{\sin\alpha}$. Ans. $x = \cot\frac{1}{2}\alpha$ or $\tan\frac{1}{2}\alpha$.

4. $x^2 + 1 = 2x\sec\alpha$. Ans. $x = \dfrac{1 \pm \sin\alpha}{\cos\alpha}$.

5. $1 - x^2 = 2x\cot\alpha$. Ans. $x = \tan\frac{1}{2}\alpha$ or $-\cot\frac{1}{2}\alpha$.

6. $x^2 - 1 = 2x\tan\alpha$. Ans. $x = \dfrac{\sin\alpha \pm 1}{\cos\alpha}$.

Find θ from the equations:

7. $27.615\cos\theta - 23.208\sin\theta = 19.094$.

8. $3.6003\sin(\theta - 8°) + 5.907\sin(\theta + 54°) = 2.6253$.

Reduce the first members by Prob. IV.

9. $a\sin\theta + b\cos\theta + c = 0$.

From this last equation find $\sin\theta$, $\cos\theta$, and $\tan\theta$ by separate quadratic equations.

CHAPTER VI.

SOLUTION OF TRIANGLES IN GENERAL.

54. A plane triangle has six parts, three sides and three angles. Of these parts the three angles are not independent of each other, because when two angles are given the third may be found from the condition that the sum of the three angles is 180°. Hence, if two angles are given, the case is the same as if all three were given.

When any three independent parts are given the remaining three may be found, but in order to be independent one of the three given parts must be a side.

55. The fact that the sum of the three angles of a plane triangle is 180° enables us to express a trigonometric function of any one angle as a similar function of the sum of the other two angles. It has been shown that

$$\begin{aligned} \sin x &= \sin(180° - x); \\ \cos x &= -\cos(180° - x); \\ \tan x &= -\tan(180° - x); \\ \cot x &= -\cot(180° - x); \\ \sec x &= -\sec(180° - x); \\ \operatorname{cosec} x &= -\operatorname{cosec}(180° - x). \end{aligned}$$

If α, β, and γ are the three angles of a triangle, we have

$$\left.\begin{aligned} \alpha &= 180° - (\beta + \gamma); \\ \beta &= 180° - (\gamma + \alpha); \\ \gamma &= 180° - (\alpha + \beta). \end{aligned}\right\} \qquad (a)$$

Therefore

$$\left.\begin{aligned} \sin\alpha &= \sin(\beta + \gamma); \\ \sin\beta &= \sin(\gamma + \alpha); \\ \sin\gamma &= \sin(\alpha + \beta); \\ \cos\alpha &= -\cos(\beta + \gamma); \end{aligned}\right\} \qquad (1)$$

$$\text{etc.} \qquad \text{etc.}$$

By dividing the equations (a) by 2 we find

$$\left.\begin{aligned} \tfrac{1}{2}\alpha &= 90^\circ - \tfrac{1}{2}(\beta+\gamma); \\ \tfrac{1}{2}\beta &= 90^\circ - \tfrac{1}{2}(\gamma+\alpha); \\ \tfrac{1}{2}\gamma &= 90^\circ - \tfrac{1}{2}(\alpha+\beta). \end{aligned}\right\} \qquad (b)$$

Therefore

$$\left.\begin{aligned} \sin\tfrac{1}{2}\alpha &= \cos\tfrac{1}{2}(\beta+\gamma); \\ \cos\tfrac{1}{2}\alpha &= \sin\tfrac{1}{2}(\beta+\gamma); \\ \tan\tfrac{1}{2}\alpha &= \cot\tfrac{1}{2}(\beta+\gamma); \\ \cot\tfrac{1}{2}\alpha &= \tan\tfrac{1}{2}(\beta+\gamma); \\ \text{etc.} &\qquad \text{etc.} \end{aligned}\right\} \qquad (2)$$

From what has been said the given parts may be

One side and the angles;
Two sides and one angle;
The three sides.

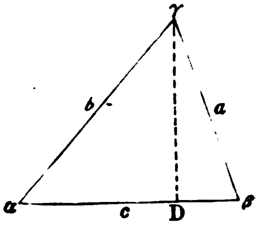
Also, when the sides and one angle are given, this angle may be either that between the given sides or opposite one of them. Hence there are four cases in all to be considered.

56. Case I. Given the angles and one side.

Theorem I. *The sides of a plane triangle are proportional to the sines of their opposite angles.*

Proof. Put

a, b, c, the three sides;
α, β, γ, their opposite angles.



From one angle, as γ, drop a perpendicular γD upon the opposite side, c. Then

$$\gamma D = b\sin\alpha; \quad \gamma D = a\sin\beta. \quad (\S\, 35)$$

Therefore

$$a\sin\beta = b\sin\alpha.$$

Dividing by $\sin\alpha\sin\beta$,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}.$$

By dropping a perpendicular from α upon a we should find, in the same way,

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Therefore $$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}, \qquad (3)$$

or $$a : b : c = \sin\alpha : \sin\beta : \sin\gamma. \quad \text{Q.E.D.} \qquad (4)$$

Def. The common value of the three quotients $\frac{a}{\sin\alpha}$, $\frac{b}{\sin\beta}$, and $\frac{c}{\sin\gamma}$ is called the **modulus** of the triangle.

THEOREM II. *The modulus of the triangle is equal to the diameter of the circumscribed circle.*

This theorem may be demonstrated by the student from the property that an inscribed angle is measured by one half the arc on which it stands.

Theorem I. enables us, when the angles and one side of any plane triangle are given, to find the remaining two sides. If the side given is c, we have

$$\left.\begin{aligned} a &= \frac{c\sin\alpha}{\sin\gamma}; \\ b &= \frac{c\sin\beta}{\sin\gamma}. \end{aligned}\right\} \qquad (5)$$

or, putting M for the modulus, we have

$$\left.\begin{aligned} a &= M\sin\alpha; \\ b &= M\sin\beta; \\ c &= M\sin\gamma. \end{aligned}\right\} \qquad (6)$$

If we put p, p', and p'' for the lengths of the perpendiculars from α, β, and γ respectively upon the opposite sides, we find, from the preceding figure,

$$\left.\begin{aligned} p &= b\sin\gamma = c\sin\beta; \\ p' &= c\sin\alpha = a\sin\gamma; \\ p'' &= b\sin\alpha = a\sin\beta. \end{aligned}\right\} \qquad (7)$$

By these equations we may find the lengths of the perpendiculars.

EXERCISES.

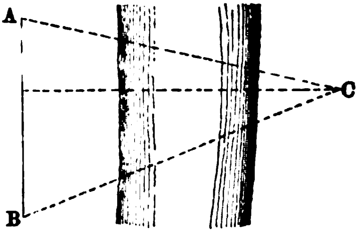
1. Given $\alpha = 78° 23'.2$, $\beta = 52° 16'.3$, $a = 796.25$; find b and c.
2. " $\alpha = 5° 26'.2$, $\beta = 72° 36'.8$, $b = 19.263$; " b and c.
3. " $\alpha = 50° 58'.7$, $\beta = 32° 50'.8$, $c = 169.37$; " a and b.

4. In order to find the distance of a point C across a river from the points A and B, a surveyor measured a base line AB and found it to be 829.72 metres. Placing a theodolite at A, he found

$$\text{Angle } BAC = 82^\circ\ 37'.6.$$

Carrying his theodolite to B, he found

$$\text{Angle } ABC = 70^\circ\ 3'.3.$$



Required the distances AC and BC across the river, and the length of the perpendicular from C on AB.

5. In a triangle ABC the angle B exceeds the angle A by 10°, the angle C exceeds the angle B by 20°, and the side AC is 2.72 904 metres. Find the angles and sides of the triangle and the length of the perpendiculars from the angles upon the opposite sides.

6. The base of an isosceles triangle is 132.643 metres, and the angle at the vertex is $32^\circ\ 53'.7$. Find the sides and the altitude.

7. One diagonal of a parallelogram measures 23 metres, and it makes angles of $32^\circ\ 17'$ and $63^\circ\ 24'$ with the sides. Find the lengths of the sides and the angles of the parallelogram.

8. From a point at a distance a from the centre of a circle of radius r tangents are drawn to the circle. Express the lengths of the tangents and the distance between the points of tangency, and compute the result when $r = 7$, $a = 12$.

57. Case II. Given two sides and the angle opposite one of them.

Let the given parts be a, b, α. We then compute the parts β, γ, and c by the formulæ (a) and (3) already found.

$$\left.\begin{aligned} \sin\beta &= \frac{b}{a}\sin\alpha; \\ \gamma &= 180^\circ - (\alpha + \beta); \\ c &= \frac{b\sin\gamma}{\sin\beta} = \frac{a\sin\gamma}{\sin\alpha}. \end{aligned}\right\} \qquad (8)$$

This case may have two solutions, as is shown in geometry. The two solutions are found in the above equations, because to a given value of $\sin\beta$ corresponds either of two angles β (§ 21), which will be supplements of each other.

But if one solution gives $\alpha + \beta > 180°$ it is not admissible, and only the lesser value is used to give the triangle. This will be the case when $a > b$.

It may also happen that $\sin\beta = \frac{b}{a}\sin\alpha$ comes out greater than unity. There is then no possible triangle which fulfils the conditions.

Example. Given $a = 152.08$, $b = 236.74$, $\alpha = 32° 29'.6$; find the remaining parts.

$\log b$,	2.374 27		
$\log\sin\alpha$,	9.730 14		
$\operatorname{colog} a$,	7.817 93		
$\log\sin\beta$,	9.922 34		
β,	$56° 44'.9$	or	$123° 15'.1$
α,	$32° 29'.6$		$32° 29'.6$
$\alpha + \beta$,	$89° 14'.5$		$155° 44'.7$
	$180° 00'.0$		$180° 00'.0$
γ,	$90° 45'.5$	or	$24° 15'.3$
$\log\sin\gamma$,	9.999 96	or	9.613 62
$\log\frac{b}{\sin\beta}$,	2.451 93		2.451 93
$\log c$,	2.451 89	or	2.065 55
c,	283.07	or	116.29

Exercises.

1. Given $a = 24$, $b = 33$, $\alpha = 31° 28'$; find the remaining parts.
2. “ $a = 34$, $b = 35.79$, $\beta = 17° 59'$; “ “ “
3. “ $a = 29$, $b = 34$, $\alpha = 30° 20'$; “ “ “
4. “ $b = 19$, $c = 18$, $\gamma = 15° 49'$; “ “ “
5. “ $a = 24$, $c = 13$, $\alpha = 115° 0'$; “ “ “

6. From a point P at a distance a from the centre of a circle of radius r a line is drawn, making the angle β with the line from P to the centre of the circle. At what distances from P will the

line intersect the circle? Compute the distances when $r = 72$, $a = 98$, and $\beta = 28° 56'$.

7. Show that if, in the present case, we take the side which is not given as the base, we can find the altitude of the triangle immediately, and afterwards may find the three required parts of the triangle from the altitude.

CASE III. **Given the three sides.**

THEOREM III. *In a triangle the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these two sides into the cosine of the angle included by them.*

In symbolic language this theorem is expressed in any of the forms

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha, \\ \text{or}\quad b^2 &= a^2 + c^2 - 2ac\cos\beta, \\ \text{or}\quad c^2 &= a^2 + b^2 - 2ab\cos\gamma. \end{aligned}\right\} \qquad (9)$$

Proof. It is shown in geometry that in any triangle the side opposite an acute angle is greater than the sum of the squares of the other two sides by twice the product of one of these sides into the projection of the other side upon it.

If α be the acute angle, we have, by this theorem,

$$a^2 = b^2 + c^2 - 2b \times (\text{projection of } c \text{ on } b).$$

By § 35, II., and the definition of projections.

$$\text{Projection of } c \text{ on } b = c\cos\alpha;$$

substituting this value of the projection,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha. \quad \text{Q.E.D.}$$

The other equations may be proved in exactly the same way.

If the containing angle is obtuse, the square of the opposite side will be greater than the sum of the squares of the containing sides. But this case is included in the trigonometric formula, because then $\cos\alpha$ is negative and $-bc\cos\alpha$ is positive. Hence the formula is applicable to all cases when regard is had to the algebraic sign.

From the first of equations (9) we obtain

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}, \qquad (10)$$

which with the two companion formulæ enable us to find the angles when the three sides are given.

59. If the angle is small, it cannot be accurately determined by means of its cosine; we therefore transform the expression as follows:

Subtracting each member of the equation from unity, we have

$$1 - \cos\alpha = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}.$$

But $1 - \cos\alpha = 2\sin^2\tfrac{1}{2}\alpha$ (§ 44), and $2bc - b^2 - c^2 = -(b-c)^2$. Therefore

$$2\sin^2\tfrac{1}{2}\alpha = \frac{a^2 - (b-c)^2}{2bc} = \frac{(a+b-c)\,(a-b+c)}{2bc}.$$

Let us now put s for half the sum of the three sides, so that

$$s = \tfrac{1}{2}(a+b+c).$$

Then
$$a + b - c = 2s - 2c;$$
$$a - b + c = 2s - 2b;$$

and the preceding equation reduces to

$$\sin^2\tfrac{1}{2}\alpha = \frac{(s-b)(s-c)}{bc}. \qquad (11)$$

The expressions for the other two angles are obtained by the same process, the letters a, b, c and α, β, γ being permuted in the orders b, c, a; β, γ, α and c, a, b; γ, α, β. We thus find

$$\left.\begin{aligned}\sin^2\tfrac{1}{2}\beta &= \frac{(s-c)(s-a)}{ca};\\ \sin^2\tfrac{1}{2}\gamma &= \frac{(s-a)(s-b)}{ab}.\end{aligned}\right\} \qquad (12)$$

These equations answer our purpose, but in determining an angle the tangent is the function to be preferred, because an angle can be determined more accurately from its tangent than from its sine or cosine. To obtain expressions for the tangent add unity to both sides of the equation (10). We then have

$$1 + \cos\alpha = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + 2bc + c^2 - a^2}{2bc}.$$

Since $1 + \cos\alpha = 2\cos^2\tfrac{1}{2}\alpha$ (§ 44), this equation reduces to

$$2\cos^2\tfrac{1}{2}\alpha = \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}.$$

Whence $$\cos^2 \tfrac{1}{2}\alpha = \frac{s(s-a)}{bc}. \tag{13}$$

Dividing (11) by this equation, and writing the corresponding equations for the other angles, we find

$$\left.\begin{aligned} \tan^2 \tfrac{1}{2}\alpha &= \frac{(s-b)(s-c)}{s(s-a)}; \\ \tan^2 \tfrac{1}{2}\beta &= \frac{(s-c)(s-a)}{s(s-b)}; \\ \tan^2 \tfrac{1}{2}\gamma &= \frac{(s-a)(s-b)}{s(s-c)}. \end{aligned}\right\} \tag{14}$$

The computation will be simplified a little by computing

$$H = \sqrt{\frac{(s-a)\,(s-b)\,(s-c)}{s}}.$$

We shall then have

$$\left.\begin{aligned} \tan \tfrac{1}{2}\alpha &= \frac{H}{(s-a)}; \\ \tan \tfrac{1}{2}\beta &= \frac{H}{(s-b)}; \\ \tan \tfrac{1}{2}\gamma &= \frac{H}{(s-c)}. \end{aligned}\right\} \tag{15}$$

By means of these equations we may compute two of the angles, and find the third by subtracting their sum from 180°. But in practice it is better to compute the three angles independently, and check the accuracy of the work by taking their sum.

If this sum comes out materially different from 180°, there is some mistake in the work; if not, it may be presumed correct.

Example. Given $a = 273.960$, $b = 198.632$, $c = 236.914$; find the angles.

$a = 273.960$		
$b = 198.632$	$\log(s-a)$, 1.907 37	
$c = \underline{236.914}$	$\log(s-b)$, 2.193 46	$\log\tan\frac{1}{2}\alpha$, 9.903 73
$2s = 709.506$	$\log(s-c)$, $\underline{2.071\ 29}$	$\log\tan\frac{1}{2}\beta$, 9.617 64
$s = 354.753$	sum of logs, 6.172 12	$\log\tan\frac{1}{2}\gamma$, 9.739 81
$s-a = 80.793$	$\log s$, $\underline{2.549\ 92}$	
$s-b = 156.121$	$\log H^2$, 3.622 20	
$s-c = 117.839$	$\log H$, 1.811 10	

$$\begin{array}{llrl} \tfrac{1}{2}\alpha, & 38^\circ\,42'.1; & \alpha = & 77^\circ\,24'.2 \\ \tfrac{1}{2}\beta, & 22^\circ\,31'.2; & \beta = & 45^\circ\ \ 2'.4 \\ \tfrac{1}{2}\gamma, & 28^\circ\,46'.8; & \gamma = & \underline{57^\circ\,33'.6} \\ & & \text{Sum} = & 180^\circ\ \ 0'.2 \quad \text{(Check.)} \end{array}$$

The discrepancy of 0'.2 is the result of the unavoidable errors from the omission of the decimals of the logarithms beyond the fifth.

Another check on the accuracy of the work is obtained by computing the modulus of the triangle from its three separate expressions (§ 56, 3), and noting whether they agree, thus:

	log a, 2.437 69	log b, 2.298 05	log c, 2.374 59
	sin α, 9.989 42	sin β, 9.849 79	sin γ, 9.926 32
log modulus,	2.448 27	2.448 26	2.448 27

The three results agree within the unavoidable limits of error.

EXERCISES.

1. Given $a = 3$, $b = 4$, $c = 5$; find the angles.

2. " $a = 37\,593$, $b = 29\,867$, $c = 40\,005$; " "

3. " $a = 2.796\,1$, $b = 23.928$, $c = 25.046$; " "

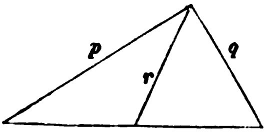
4. The base of a parallelogram is 13, each side is 6, and its lesser diagonal is 12. Find its angles.

5. If the sides of a parallelogram are a and b, and one diagonal is p, express its angles.

6. The parallel sides of a trapezoid are 12 and 17, and the non-parallel sides 6 and 7. Find its four angles.

Suggestion. Divide the trapezoid into a triangle and a parallelogram.

7. In a triangle are given the two sides, p and q, and the medial line r from the vertex to the middle point of the base. Prove Base $= \sqrt{2p^2 + 2q^2 - 4r^2}$.



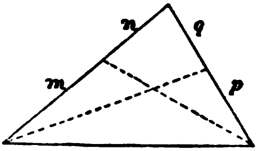
8. Given the three medial lines r, r', r'' of a triangle; find the sides of the triangle from the preceding result.

Ans. $\tfrac{2}{3}\sqrt{2r^2 + 2r'^2 - r''^2}$; $\tfrac{2}{3}\sqrt{2r^2 + 2r''^2 - r'^2}$; $\tfrac{2}{3}\sqrt{2r'^2 + 2r''^2 - r^2}$.

9. Of the bisectors of the angles at the base of a triangle, the one cuts the opposite side in the ratio $m : n$, and the other in the

ratio $p : q$. By means of the equation (10) express the cosine of the angle at the vertex of the triangle.

$$\text{Ans. } \cos\alpha = \frac{p^2n^2 + q^2m^2 - p^2m^2}{2pqmn}.$$



Note. In the solution of this question apply the theorem of geometry which defines the ratio in which the bisector of an angle cuts the opposite side.

60. Case IV. **Given two sides and the included angle.**

This case may be readily solved by Theorem III., because if the given sides are b and c, and the given included angle is α, we have for the third side

$$a = \sqrt{b^2 + c^2 - 2bc\cos\alpha}.$$

Then, having the three sides, the remaining angles may be found as in the last section. But there is a more convenient method founded on the following theorem:

Theorem IV. *As the sum of any two sides*
is to their difference,
so is the tangent of half the sum of the angles opposite these sides
to the tangent of half their difference.

Proof. From the equation

$$b : c :: \sin\beta : \sin\gamma, \qquad \text{(Th. I.)}$$

we have, by composition and division,

$$b + c : b - c :: \sin\beta + \sin\gamma : \sin\beta - \sin\gamma.$$

But
$$\sin\beta + \sin\gamma = 2\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta-\gamma); \qquad (\S\,44)$$
$$\sin\beta - \sin\gamma = 2\cos\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\beta-\gamma).$$

Substituting these values, and expressing the proportion as a fraction,

$$\frac{b+c}{b-c} = \frac{\sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\gamma)}{\cos\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\beta-\gamma)}$$
$$= \tan\tfrac{1}{2}(\beta+\gamma)\cot\tfrac{1}{2}(\beta-\gamma) \qquad (\S\,33)$$
$$= \frac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)}.$$

Therefore

$$b + c : b - c :: \tan\tfrac{1}{2}(\beta+\gamma) : \tan\tfrac{1}{2}(\beta-\gamma). \quad \text{Q.E.D.} \qquad (16)$$

The solution is now obtained as follows: We have

$$\tfrac{1}{2}(\beta+\gamma) = 90^\circ - \tfrac{1}{2}\alpha;$$

$$\tan\tfrac{1}{2}(\beta+\gamma) = \cot\tfrac{1}{2}\alpha. \qquad (\S\,55)$$

Because the angle α is given, the only unknown term of the proportion is $\tan\frac{1}{2}(\beta-\gamma)$. This is given by the equation

$$\tan\tfrac{1}{2}(\beta-\gamma) = \frac{b-c}{b+c}\tan\tfrac{1}{2}(\beta+\gamma), \qquad (17)$$

which is derived from the proportion (15).

By this equation we obtain $\frac{1}{2}(\beta-\gamma)$, which being added to and subtracted from $\frac{1}{2}(\beta+\gamma)$ gives β and γ. The remaining side of the triangle may then be found by Case I. But when this side as well as the angles are required, a more elegant method may be followed, which will be explained in the next section.

Example. Given $b = 4.567$, $c = 3.456$, $\alpha = 56^\circ\ 7'.8$; find the remaining parts.

$$\begin{array}{rl} b, & 4.567 \\ c, & \underline{3.456} \\ b-c, & 1.111 \\ b+c, & 8.023 \\ \log(b-c), & 0.045\ 71 \\ \text{colog}\,(b+c), & 9.095\ 66 \\ \log\tan\frac{1}{2}(\beta+\gamma), & \underline{0.273\ 14} \\ \log\tan\frac{1}{2}(\beta-\gamma), & \underline{\underline{9.414\ 51}} \end{array}$$

$$\begin{array}{rl} & 180^\circ \\ \alpha, & \underline{56^\circ\ \ 7'.8} \\ \beta+\gamma, & 123^\circ\ 52'.2 \\ \frac{1}{2}(\beta+\gamma), & 61^\circ\ 56'.1 \\ \frac{1}{2}(\beta-\gamma), & \underline{14^\circ\ 33'.6} \\ \beta, & 76^\circ\ 29'.7 \\ \gamma, & \underline{\underline{47^\circ\ 22'.5}} \end{array}$$

$$\begin{array}{rl} \log b, & 0.659\ 63 \\ \log\sin\beta, & \underline{9.987\ 82} \\ \log\text{Mod.}, & 0.671\ 81 \\ a = & 3.899\ 8 \end{array}$$

$$\begin{array}{rl} \log c, & 0.538\ 57 \\ \log\sin\gamma, & \underline{9.866\ 76} \\ \log\text{Mod.}, & 0.671\ 81 \\ \log\sin\alpha, & \underline{9.919\ 23} \\ \log a, & 0.591\ 04 \end{array}$$

EXERCISES.

1. Given $a = 12.34$, $b = 43.21$, $\gamma = 34^\circ\ 12'$; find rem. angles.
2. " $b = \sqrt{5}$, $c = \sqrt{3}$, $\alpha = 35^\circ\ 53'$; " "
3. " $a = 35.79$, $c = 1.246\,8$, $\beta = 97^\circ\ 53'$; " "
4. " $a = 189$, $b = 114.75$, $\gamma = 107^\circ\ 48'$; " "

61. If in the present case all three remaining parts are wanted, formulæ may be derived as follows:

From the equations (3) we derive

$$\frac{b}{a}=\frac{\sin\beta}{\sin\alpha}; \qquad \frac{c}{a}=\frac{\sin\gamma}{\sin\alpha}. \qquad (a)$$

Adding these equations we have

$$\frac{b+c}{a}=\frac{\sin\beta+\sin\gamma}{\sin\alpha}=\frac{2\sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\gamma)}{\sin\alpha} \qquad (\S\,44)$$

$$=\frac{2\cos\frac{1}{2}\alpha\cos\frac{1}{2}(\beta-\gamma)}{2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha} \qquad (\S\,55)$$

$$=\frac{\cos\frac{1}{2}(\beta-\gamma)}{\sin\frac{1}{2}\alpha}. \qquad (b)$$

Subtracting the equations (*a*) we have

$$\frac{b-c}{a}=\frac{\sin\beta-\sin\gamma}{\sin\alpha}=\frac{2\cos\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\beta-\gamma)}{2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}$$

$$=\frac{\sin\frac{1}{2}(\beta-\gamma)}{\cos\frac{1}{2}\alpha}. \qquad (c)$$

From the equations (*c*) and (*b*) we obtain

$$\left.\begin{aligned} a\sin\tfrac{1}{2}(\beta-\gamma)&=(b-c)\cos\tfrac{1}{2}\alpha,\\ a\cos\tfrac{1}{2}(\beta-\gamma)&=(b+c)\sin\tfrac{1}{2}\alpha;\end{aligned}\right\} \qquad (18)$$

which equations are readily solved by Prob. I. Chap. V.

By taking the quotient of these equations we may readily deduce the relation (16).

EXERCISES.

1. Given $b=2956.2$, $c=9090.8$, $\alpha=98°\ 29'.6$; find β, γ, and a.

2. A surveyor lays off two lines from the same point: the one due north, 279.25 metres, the other east 15° north, 109.262 metres. How far apart are the ends of the lines, and what is the direction of the line joining them?

3. The sides of a parallelogram are 26 and 15, and one angle is 126° 52′.2. Find the lengths of the two diagonals and the angles which they make with the sides.

4. Given the two diagonals d and d' of a parallelogram and the angle ε which they form; express the sides and angles of the parallelogram algebraically and compute them for the special case $d=5$, $d'=6$, $\varepsilon=49°\ 18'$.

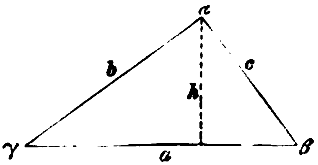
Areas of Triangles.

62. Theorem V. *The area of a triangle is equal to half the product of any two sides by the sine of their included angle.*

Proof. It is shown in geometry that the area is half the base into the altitude. Now in the figure,

Altitude $h = b \sin \gamma$.

Therefore, a being the base,



$$\text{Area} = \tfrac{1}{2}ah = \tfrac{1}{2}ab \sin \gamma. \quad \text{Q.E.D.}$$

Cor. 1. *If two triangles have two sides of the one respectively equal to two sides of the other, and the angles which these sides form supplementary, the triangles will be equal in area.*

For the sines of the supplementary angles are equal.

Cor. 2. Since we may take any one side as a base, if we call h, h', h'' the altitudes above the respective sides a, b, and c, we shall have

$$ah = bh' = ch'',$$

and

$$ab \sin \gamma = bc \sin \alpha = ca \sin \beta.$$

For these expressions are each double the area of the triangle.

Exercises.

1. Given $a = 75$, $b = 29$, $\beta = 16° \, 15'.6$; find the remaining parts and the areas of the two triangles which may be formed.

2. Express the area of a parallelogram of which two adjoining sides are of given length, a and b, and make with each other a given angle δ.

3. Express the area of a triangle in terms of a base, c, and the two adjacent angles, α and β.

Ans. $\frac{1}{2}c^2 \dfrac{\sin \alpha \sin \beta}{\sin (\alpha + \beta)}$.

4. In a parallelogram is given a diagonal of length d, and the angles θ and φ which the diagonal makes with the two sides adjoining it. Express the area of the parallelogram.

5. Express the area of the parallelogram in terms of the lengths, d and d', of its diagonals, and the angle ε at which they intersect.

6. In a quadrilateral are given the four sides, $a = 25.63$, $b = 24.09$, $c = 9.92$, $d = 29.97$, and the angle, 78° 25′, which the sides a and b form with each other. Compute the angles and the area of the quadrilateral.

7. A triangle ABC is to be divided into two parts of equal area by a line parallel to AB. What will be the ratios of the segments into which the other two sides are divided? Ans. $\sqrt{\frac{1}{2}} : 1 - \sqrt{\frac{1}{2}}$.

8. A city lot fronts 60 feet on a street, and the parallel sides run back, the one 100 and the other 135 feet. It is to be divided into two equal parts by a line parallel to its sides. What will be the frontage of each part, and the length of the dividing line?

Remark.—The figure of the lot is a trapezoid, and the problem is that of dividing a trapezoid into two equal portions by a line parallel to the base. Let us put

a, b, the parallel sides;

$m : 1 - m$, the ratio in which the non parallel sides, and therefore the altitude, is divided by the dividing line;

k, the length of the dividing line.

The unknown quantities of the problem are then m and k. If we put h for the altitude of the entire trapezoid, the altitudes of the two parts will be mh and $(1 - m)\,h$ respectively. Therefore the areas will be

$$\frac{a+k}{2}\,mh \quad \text{and} \quad \frac{k+b}{2}\,(1-m)\,h,$$

the equality of which gives the first condition. For the second condition we have the geometrical theorem that the difference between the dividing line and either of the parallel sides is proportional to its distance from such side. This gives the proportion

$$a - k : k - b :: m : 1 - m,$$

whence $$(k - b)\,m = (a - k)\,(1 - m).$$

The quotient of this equation by the preceding one gives an equation from which m is eliminated, and from which we find the value of k.

$$k = \sqrt{\frac{a^2 + b^2}{2}}.$$

We then find for m the equation

$$m = \frac{a - k}{a - b}.$$

Applying this method to the problem under consideration, we find

$$k = 118.796 \text{ feet};$$

$$m = 0.462\,97;$$

frontages of lots, 27.778 and 32.222 feet.

CHAPTER VII.

THE THEORY OF POLYGONS.

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63. A polygon is completely determined when the positions of its vertices, taken in regular order, are given. The polygon may then be formed by joining each pair of consecutive vertices by a straight line.

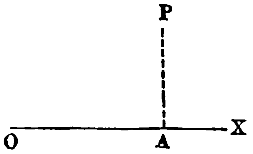
The positions of the vertices may be defined by their co-ordinates, on a system now to be explained.

64. *Co-ordinates of a point.* In geometry the position of a point is fixed by assigning to it certain lines or numbers indicating its situation relative to a fixed line, and a point on that line.

Def. Any numbers or lines which determine the position of a point are called the **co-ordinates** of that point.

Rectangular co-ordinates. Let OX be the fixed line of reference, and O a point of reference on that line from which we measure.

Let P be a point whose position is to be expressed. From P drop a perpendicular PA upon OX. Then:



The line PA is called the **ordinate** of the point P.

The line OA is called the **abscissa** of the point P.

The ordinate and abscissa are called **rectangular co-ordinates** of the point.

The indefinite line OX along which the abscissas are measured is called the **axis of abscissas** or the **axis of X.**

The zero point, O from which the co-ordinates are measured is called the **origin.**

When the rectangular co-ordinates are given the position of the point is completely determined.

To find the point when the ordinate and abscissa are given, we measure off from O, on the line OX, a distance equal to the given abscissa.

At the end of this distance we erect a perpendicular equal to the given ordinate.

The end of this perpendicular will be the required position of the point.

The abscissa of a point is represented by the symbol x, the ordinate by the symbol y.

If x is positive, its length is laid off from O toward the right; if negative, toward the left

If the ordinate y is positive, it is measured upward from the axis of X; if negative, downward.

Exercises.

Draw a line OX 4 or 5 inches long as a line of reference, and lay off points having the following co-ordinates from a zero point near the middle of the line:

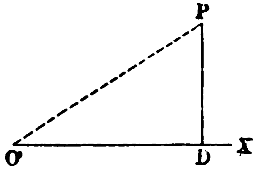
1. $x = 1$, $y = 2$ inches;
2. $x = 2$, $y = 1$ "
3. $x = 2$, $y = 2$ "
4. $x = 1$, $y = 1$ "
5. $x = 1$, $y = -2$ "
6. $x = 2$, $y = -1$ "
7. $x = -2$, $y = -2$ "
8. $x = -1$, $y = 2$ "
9. $x = 0$, $y = -2$ "

Polar co-ordinates. Draw a line from O to P, and call r its length and φ the angle XOP. Then

$$OD = x = r \cos \varphi,$$

$$DP = y = r \sin \varphi,$$

and Problem I. of Chapter V. reduces to:



Given the co-ordinates x and y of a point, to find the distance and direction of the point from the origin.

Since the quantities r and φ completely determine the position of P, they are also co-ordinates. To distinguish them they are called **polar co-ordinates.**

EXERCISE.

Eight points have the following several co-ordinates. Find the values of φ and r (the values of r being all equal), and note what relation exists among the values of φ.

1. $x = +4, \quad y = +3;$
2. $x = +3, \quad y = +4;$
3. $x = -3, \quad y = -4;$
4. $x = -4, \quad y = +3;$
5. $x = -4, \quad y = -3;$
6. $x = -3, \quad y = -4;$
7. $x = +3, \quad y = -4;$
8. $x = +4, \quad y = -3.$

65. *Definition of direction of lines and angle between them.* Two finite lines which do not meet are considered to form a certain angle with each other; namely, the angle which would be formed if they were continued until they met, or if a line parallel to the one were drawn through any point of the other.

Since at the point of crossing four angles are formed, we may, in the absence of any convention, regard either of these angles as that between the lines. But as opposite angles are equal, these angles only have two *different* values.

Also, in the absence of a convention, we may regard any angle as either positive or negative. Hence to a given inclination of the lines may be assigned any one of four different values, which values are divisible into two supplementary pairs.

Example. If two lines intersect at an angle of 85°, we may consider their inclination, or the angle which they form, to be

$$\begin{aligned} &\text{either} \quad 85^\circ, \\ &\text{or} \quad 95^\circ, \\ &\text{or} \quad -85^\circ = 275^\circ, \\ &\text{or} \quad -95^\circ = 265^\circ. \end{aligned}$$

This ambiguity is avoided by the following conventions:

1. We assign to each line a positive and a negative direction. The positive direction is that from the beginning to the end of a finite line. The angle they form is then that between their *positive directions*. This is the same as the angle between two lines going out from the same point in the respective positive directions of the lines.

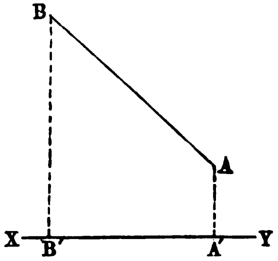
2. We consider one side of the angle as that from which we measure, and we measure the angle to the other side in a positive direction, as explained in §§ 3 and 8.

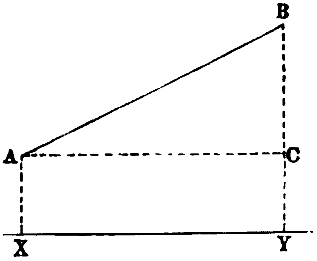
The four values are thus reduced to one.

If two lines are parallel, their angle is 0° or 180° according as they are *similarly directed* or *oppositely directed.*

Projections of Lines.

66. *Def.* The projection of a finite straight line AB upon an indefinite line XY is the distance $A'B'$ between the feet of the perpendiculars from A and B upon the indefinite line.





To find the length of a projection. Through one end of the line, as A, draw AC parallel to XY, meeting BY in C. Then

$$AC = XY;$$
$$AC = AB \cos BAC.$$

Hence

$$\text{Projection } XY = AB \cos BAC. \qquad (1)$$

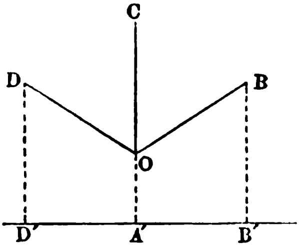
That is,

The projection is equal to the length of the line projected into the cosine of the angle which the two lines form with each other.

Algebraic sign of a projection. Let the positive direction of the line OB be from O to B, and let the line turn on O into the successive positions OC and OD.

If on the line of projection we regard directions toward the right as positive, the projection $A'B'$ will be positive.

The whole line OC will be projected at the point C; the projection will therefore be zero, a result given by the formula (1), because cos 90° $= 0$.

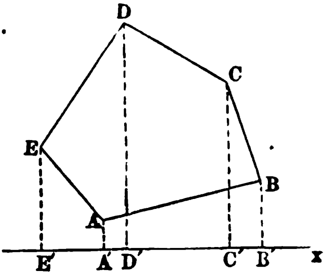


The projection $A'D'$ of OD will be negative because it falls in the negative direction. This also corresponds to the formula, because the angle between the two directions OD and $A'B'$ is obtuse.

If we suppose OB to perform a complete revolution around O, we readily see that its projection goes through the same series of changes as the cosine of the angle which it forms with the line of projection.

67. *Projection of sides of a polygon.* Let $ABCDE$ be any polygon the positive directions of whose sides correspond with the circuit we should form in going round the polygon, so as to reach its vertices in alphabetical order. We shall then have, for the projections on the line X,

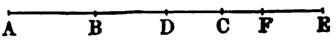
Proj. of $AB = A'B'$, which is $+$;
" $BC = B'C'$, " $-$;
" $CD = C'D'$, " $-$;
" $DE = D'E'$, " $-$;
" $EA = E'A'$, " $+$.



The positive direction being arbitrary, we might equally take the directions AE, ED, DC, etc., as positive. Each of the projections would then have the opposite algebraic sign from that just given.

The student will remark that the projection of the line is positive or negative according as the projection of its end is on the positive or negative side of the projection of its beginning. We wish now to determine the sum of the projections, and for this purpose must understand the algebraic addition of lines.

68. *Algebraic addition of lines.* In geometry the *sum* of two segments AB and BC is defined as the segment AC, formed by putting AB and BC end to end in the same straight line.



In trigonometry and modern geometry we distinguish between the beginning and the end of each segment, and between the positive and negative directions upon the segment; the *positive* direction being *from* the beginning *toward* the end; the *negative*, *from* the end *toward* the beginning.

When this distinction is attended to we must, in designating a segment by letters at its termini, write that letter first which is at the beginning of the segment, so that the letters shall follow each other in the positive order. The segment BA will then be the negative of the segment AB.

We now generalize the definition of the addition of lines as follows:

Def. The algebraic sum of several lines is formed by placing the beginning of each line after the first at the end of the line next preceding. This sum is then the segment from the beginning of the first line to the end of the last one.

Example.—In the preceding figure we have

$$AC + CE = AE,$$

as in geometry, because both segments are positive.

But if we consider the segment CD as beginning at C and ending at D, then, by the above definition, the algebraic sum of the segments AC and CD will be the segment AD, from the beginning of AC to the end of CD. That is, a negative segment will be subtractive in the same way that in algebra the addition of a negative to a positive quantity implies subtraction.

In general, whenever A, B, C, D, E, F represent points upon a straight line, we have

$$AB + BC + CD + DE + EF = AF,$$

however these points may be situated.

If the end of the last line coincides with the beginning of the first, the sum will be zero, by definition. Hence, however the points A, B, and C may be situated, we have

$$AB + BA = 0;$$
$$AB + BC + CA = 0.$$

69. Let us now return to the projected polygon. On the preceding system, the sum of the projections of the several sides upon the line X is

$$A'B' + B'C' + C'D' + D'E' + E'A' = 0.$$

The same thing would be true if we took any other straight line as the line of projection. Hence:

Theorem I. *The algebraic sum of the projections of the sides of a polygon upon any straight line is zero.*

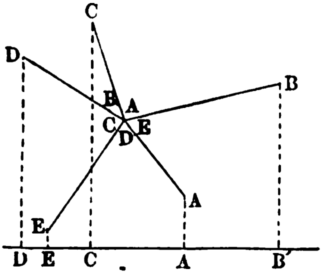
Since the projection of each side is equal to its length into the cosine of the angle it makes with the line of projection, this theorem may be expressed in the following form:

If the sides of a polygon be a, b, c, etc., and the angles which these sides make with any straight line be α, β, γ, etc., we shall have

$$a \cos \alpha + b \cos \beta + c \cos \gamma + \text{etc.} = 0.$$

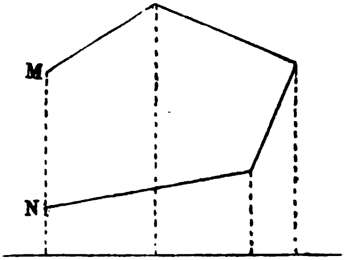
We may imagine the sides of the polygon all taken up and placed with their beginnings at the same point, their length and direction remaining unchanged.

Their several projections will then have the same values as before, and in consequence the algebraic sum of the projections will still be zero.



70. We now have the following theorem, the demonstration of which is left as an exercise for the student:

Theorem II. *If the algebraic sums of the projections of three or more straight lines upon any two non-parallel lines are each zero, these lines when placed end to end without changing their directions will form a polygon, the end of the last line falling upon the beginning of the first.*



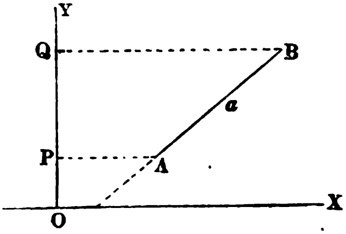
Note. To fix the ideas the student may suppose the lines as first given to all emanate from one point, as in the last figure.

The demonstration is begun by showing that in case the sum of the projections upon a straight line is zero, then, when the lines are placed end to end, the end of the last line and the beginning of the first must lie on the same perpendicular to the line of projection. Thus, in the figure, the sum of the projections of the four unbroken lines is zero, although they do not form a polygon. But, with such a figure, the projections will not be zero on any other non-parallel line.

71. Theorem III. *If a, b, c, etc., be the sides of a polygon, and α, β, γ, etc., the angles which these sides form with any straight line, we shall have*

$$a \sin \alpha + b \sin \beta + c \sin \gamma + \text{etc.} = 0.$$

Proof. Let OX be the base line from which the angles α, β, γ, etc., are counted; AB, any side of the polygon; a, its length; α, the angle which AB makes with X.



Draw OY perpendicular to OX, and let PQ be the projection of AB upon OY. AB will then make with OY an angle $90° - \alpha$, and we shall have

$$\text{Projection } PQ = a \cos (90° - \alpha) = a \sin \alpha.$$

Treating all the other sides in the same way, the algebraic sum of their projections upon OY is found to be

$$a \sin \alpha + b \sin \beta + c \sin \gamma + \text{etc.},$$

which sum is zero by Theorem I.

Theorem IV. *If the sum of the projections of a series of straight lines upon any two non-parallel lines be zero, the sum of their projections upon any third line will be zero.*

Note. This theorem follows immediately from Theorems I. and II., but we prove it algebraically in order to show an elegant application of the addition theorem.

Proof. Put

α, β, γ, etc., the angles which the straight lines make with one of the lines of projection;

π, the angle which the first two lines of projection make with each other.

Then $\alpha - \pi$, $\beta - \pi$, $\gamma - \pi$, etc., will be the angles which the lines make with the second line of projection.

By hypothesis we have

$$a \cos \alpha + b \cos \beta + c \cos \gamma + \text{etc.} = 0; \qquad (a)$$

$$a \cos (\alpha - \pi) + b \cos (\beta - \pi) + c \cos (\gamma - \pi) + \text{etc.} = 0.$$

The last equation, by the addition theorem, reduces to

$$\cos \pi (a \cos \alpha + b \cos \beta + c \cos \gamma + \ldots)$$
$$+ \sin \pi (a \sin \alpha + b \sin \beta + c \sin \gamma + \ldots) = 0.$$

The first term of this equation vanishes by (a).

Hence, the whole sum being zero, the second term must also vanish, which requires that we either have

$$\sin \pi = 0,$$

which will give $\pi = 0$ or 180°,—in which case the two lines would be parallel,—or

$$a \sin \alpha + b \sin \beta + c \sin \gamma + \text{etc.} = 0. \qquad (b)$$

Since, by hypothesis, the two lines are not parallel, the equation b must hold true.

Now let θ be the angle which any third line of projection forms with the first line. The angles which the lines a, b, c, etc., form with this third line will then be

$$\alpha - \theta, \quad \beta - \theta, \quad \gamma - \theta, \quad \text{etc.}$$

Therefore the sum of the projections upon this line will be

$$a \cos (\alpha - \theta) + b \cos (\beta - \theta) + c \cos (\gamma - \theta) + \text{etc.},$$

which reduces to

$$\cos \theta (a \cos \alpha + b \cos \beta + c \cos \gamma + \text{etc.})$$
$$+ \sin \theta (a \sin \alpha + b \sin \gamma + c \sin \gamma + \text{etc.}),$$

a sum which vanishes by (a) and (b), whatever be the value of θ, thus proving the theorem.

72. *Cor.* From §§ 69 and 71 it follows that if all the sides of a polygon but one are given in length and direction,—a, b, c, etc., being the lengths, and α, β, γ, etc., the angles with a fixed base line,—and if l be the omitted side and ς its angle, we shall have

$$\left.\begin{aligned} l \sin \varsigma &= -a \sin \alpha - b \sin \beta - c \sin \gamma - \text{etc.}; \\ l \cos \varsigma &= -a \cos \alpha - b \cos \gamma - c \cos \gamma - \text{etc.}; \end{aligned}\right\} \qquad (2)$$

which will determine l and ς, by Prob. I. Chap. V.

Example. A surveyor measures off courses and distances as follows:

I. North 80° 28′ east, 42.68 metres.
II. North 23° 22′ east, 22.79 "
III. North 65° 49′ west, 31.96 "
IV. South 59° 58′ west, 40.13 "

What distance and direction will carry him to his starting-point?

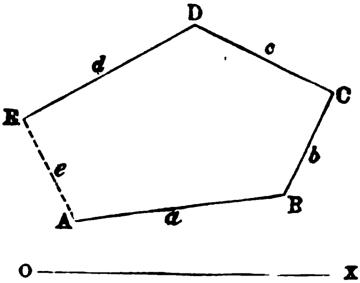
We note that the expression

North $r°$ east

means

$r°$ east of north.

Taking the east and west line OX as the base from which we measure angles, we readily find the angles made by each side with the base, as shown in the following table, which also gives the values of $a \sin \alpha$, $b \sin \beta$, etc., and $a \cos \alpha$, $b \cos \beta$, etc., as computed from the given data:



Side.	Length.	Angle.	$a \sin \alpha$, etc.	$a \cos \alpha$, etc.
a	42.68	9° 32′	+ 7.069	+ 42.090
b	22.79	66° 38′	+ 20.921	+ 9.039
c	31.96	155° 49′	+ 13.098	− 29.155
d	40.13	210° 2′	− 20.085	− 34.742
			+ 20.998	− 12.768

Hence from (2) we have, for the last side,

$$l \sin \varsigma = -20.998;$$
$$l \cos \varsigma = +12.768;$$

from which we find

$$l = 24.575;$$
$$\zeta = 301^\circ\ 18'.1.$$

Expressed in the language of surveyors, the angle ζ indicates the direction, South 31° 18′.2 East.

It will be seen that we have taken as the positive direction of the last line that from the point last reached to the starting-point. This is in accordance with the convention that the positive directions of the several sides of a polygon are so taken that in passing around it the directions shall all be positive or all negative.

But we might equally consider the problem: Having proceeded along a series of connected lines, AB, BC, etc., of which the lengths and directions are given, to E, what is our distance and direction from our starting-point A? It is evident that the distance and direction are the length and direction of the line AE, which is simply the negative of the side EA necessary to complete the polygon. If we call ε the angle of direction of AE, the equations for determining l and ε would be

$$\left.\begin{aligned} l \sin \varepsilon &= a \sin \alpha + b \sin \beta + \text{etc.};\\ l \cos \varepsilon &= a \cos \alpha + b \cos \beta, \text{ etc.}; \end{aligned}\right\} \qquad (3)$$

and, in the preceding example, we should have

$$l \sin \varepsilon = +\ 20.998;$$
$$l \cos \varepsilon = -\ 12.768;$$

which would give

$$l = 24.575;$$
$$\varepsilon = 121^\circ\ 18'.1.$$

EXERCISES.

1. A surveyor ran a course S. 12° 13′ E., 289.26 metres, and thence N. 82° 49′ E., 92.68 metres. What is his direction and distance from his starting-point? What is the direction and distance of the starting-point from him?

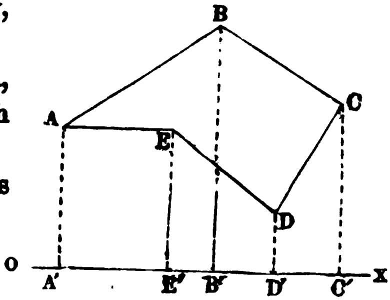
2. Five sides of an irregular hexagon taken in order have the following lengths and directions relative to a fixed line:

a.	Length,	297.43	metres;	direction,	332° 6′.8;
b.	"	606.07	"	"	222° 42′.3;
c.	"	421.02	"	"	157° 59′.8;

d. Length, 343.90 metres; direction, 5° 22′.1;
e. " 40.92 " " 125° 2′.2.

What is the length and direction of the remaining side?

73. *Areas of polygons.* When the sides of a polygon are all given in length and direction, the area may be computed by a process demonstrated in geometry, but which we shall describe here.



Let $ABCDE$ be any polygon, and OX the base line from which we measure angles.

The area of this polygon is equal to

$$\text{Area } A'ABCC'$$
$$\textit{minus} \text{ Area } A'AEDCC'.$$

The first area is equal to the sum of the areas of the two trapezoids

$$A'ABB' \text{ and } B'BCC'.$$

The second is equal to the sum of the areas of the three trapezoids

$$C'CDD', \quad D'DEE', \quad \text{and} \quad E'EAA'.$$

It will be noted that there is one trapezoid for each side of the polygon, of which the non-parallel sides are the side of the polygon and its projection upon the base line.

We have for the area of the first trapezoid, noting that the base line OX is perpendicular to the parallel sides,

$$\text{Area } A'ABB' = \tfrac{1}{2}(AA' + BB')\, A'B'.$$

Putting, as before,

a, b, c, etc., for the length of the sides AB, BC, CD, etc.;

α, β, γ, etc., for the angles which they form with OX;

putting, also,

p, the length AA',

we have

$$AA' = p;$$
$$BB' = p + a \sin \alpha;$$
$$A'B' = a \cos \alpha.$$

Substituting these values, we have, for the area of the first trapezoid,

$$\text{Area } A'ABB' = \tfrac{1}{2}(2p + a \sin \alpha)\, a \cos \alpha.$$

Passing on to the other trapezoids, we have, for the lengths of the several perpendiculars,

$$\begin{aligned}
BB' &= p + a \sin \alpha ; \\
CC' &= p + a \sin \alpha + b \sin \beta ; \\
DD' &= p + a \sin \alpha + b \sin \beta + c \sin \gamma ; \\
EE' &= p + a \sin \alpha + b \sin \beta + c \sin \gamma + d \sin \delta \\
&= p - e \sin \varepsilon ; \qquad\qquad (\S\, 71) \\
AA' &= p.
\end{aligned}$$

Also, for the altitudes of the trapezoids between their parallel sides,

$$\begin{aligned}
A'B' &= a \cos \alpha ; \\
B'C' &= b \cos \beta ; \\
C'D' &= c \cos \gamma ; \\
D'E' &= d \cos \delta ; \\
E'A' &= e \cos \varepsilon .
\end{aligned}$$

We thus have, for the several areas,

$$\left.\begin{aligned}
&\tfrac{1}{2}(2p + a \sin \alpha)\, a \cos \alpha ; \\
&\tfrac{1}{2}(2p + 2a \sin \alpha + b \sin \beta)\, b \cos \beta ; \\
&\tfrac{1}{2}(2p + 2a \sin \alpha + 2b \sin \beta + c \sin \gamma)\, c \cos \gamma ; \\
&\tfrac{1}{2}(2p + 2a \sin \alpha + 2b \sin \beta + 2c \sin \gamma + d \sin \delta)\, d \cos \delta ; \\
&\tfrac{1}{2}(2p + 2a \sin \alpha + 2b \sin \beta + 2c \sin \gamma \\
&\qquad\qquad + 2d \sin \delta + e \sin \varepsilon)\, e \cos \varepsilon .
\end{aligned}\right\} (a)$$

It has been shown that the required area of the polygon is found by subtracting the last three of these areas from the first two.

But in reaching this conclusion we took no account of algebraic signs, and so virtually considered the areas all positive. Now, as the figure is drawn, $\cos \alpha$ and $\cos \beta$ are positive, and the cosines of γ, δ, and ε are negative. Hence if we put the sign $+$ before each of the areas (a), the subtraction will be indicated by the negative character of those products which have $\cos \gamma$, $\cos \delta$, and $\cos \varepsilon$ as factors, and so the algebraic expression will be correct.

If we add up the quantities (a), beginning with the terms in p, we see that the coefficient of p is

$$a \cos \alpha + b \cos \beta + \ldots + e \cos \varepsilon ,$$

which is zero. Hence the quantity p disappears from the expression for the area. Since p is defined as the distance below A at which the base line OX is drawn, this is the same as saying that the area of the polygon is independent of this distance, which evidently must be true. In fact, if we suppose the base line OX to move up or down, remaining parallel to itself, it will add and subtract equal areas to or from the positive and negative trapezoids, and so will leave the algebraic sum of the areas unchanged.

Now let us suppose p zero, and put

$$\left.\begin{aligned}
y_1 &= a \sin \alpha;\\
y_2 &= 2a \sin \alpha + b \sin \beta = y_1 + a \sin \alpha + b \sin \beta;\\
y_3 &= 2a \sin \alpha + 2b \sin \beta + c \sin \gamma = y_2 + b \sin \beta + c \sin \gamma;\\
y_4 &= y_3 + c \sin \gamma + d \sin \delta;\\
y_5 &= y_4 + d \sin \delta + e \sin \varepsilon.
\end{aligned}\right\} (b)$$

(We remark that y_5 will come out equal to $-e \sin \varepsilon$ if everything is correct.)

We shall then have, for the double of the area of the polygon,

$$2\,\text{Area} = y_1 a \cos \alpha + y_2 b \cos \beta + y_3 c \cos \gamma + y_4 d \cos \delta + y_5 e \cos \varepsilon. \quad (4)$$

As an example, let us compute the area of the polygon investigated in the example of § 72. The following table shows the principal parts of the computation:

$a \sin \alpha$, $b \sin \beta$, etc.	Sums of Pairs.	y_1, y_2, etc.	$a \cos \alpha$, $b \cos \beta$, etc.	Products.
+ 7.069	+ 27.990	+ 7.069	+ 42.090	+ 297.5
+ 20.921	+ 34.014	+ 35.059	+ 9.039	+ 316.9
+ 13.093	− 6.992	+ 69.073	− 29.155	− 2013.9
− 20.085	− 41.083	+ 62.081	− 34.742	− 2156.8
− 20.998		+ 20.998	+ 12.768	+ 268.1
				− 3288.2

The first column gives the values of $a \sin \alpha$, $b \sin \beta$, $c \sin \gamma$, etc., already computed in the preceding example.

The second column gives the values of $a \sin \alpha + b \sin \beta$, $b \sin \beta + c \sin \gamma$, etc., which are added to each value of y to form the value of y next following, as shown in (b).

The third column gives the values of y_1, y_2, y_3, y_4, y_5, computed by the formulæ (b), from the numbers in the first two columns.

The fourth column gives the values of $a \cos \alpha$, $b \cos \beta$, $c \cos \gamma$, etc., already computed.

The fifth column gives the products which enter into the equation (c).

The algebraic sum being twice the area, we have

$$\text{Area of polygon} = 1644.1 \text{ square metres.}$$

74. It will be noticed that the area of the polygon comes out negative, and the question arises, What interpretation is to be put on this result? The answer is that the conventions of positive and negative as employed in this chapter, as applied to lines, express only the directions in which the lines are reckoned. By a change of direction the algebraic signs of all the trapezoids, and therefore of the area of the polygon itself, will be changed. A little consideration will show that this area will come out positive when we go round the polygon in what we have called the negative direction, and *vice versa.*

The algebraic sum of the areas, whether positive or negative, will always be the true area of the trapezoid under one important condition: that none of the sides cross each other. In this case the system of applying the algebraic signs will lead to the areas on the two sides of the point of crossing having opposite signs. Hence the result finally obtained will be the *difference* of the two areas.

EXERCISES.

1. If the lengths and directions of three of the four sides of a quadrilateral are

$a = 262.72$ metres; $\alpha = 39° 49'$;
$b = 109.79$ " $\beta = 150° 26'$;
$c = 300.63$ " $\gamma = 242° 52'$;

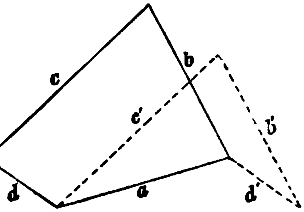
find the remaining side and the area.

2. The sides of a quadrilateral, taken in regular order, have the following lengths and directions, in part:

$a = 29$; $\alpha = 12° 26'$;
$b = 52$; $\beta = 75° 58'$;
c to be found; $\gamma = 172° 3'$;
$d = 66$; δ to be found.

Find c and δ.

3. Prove geometrically that if the sides of a polygon be joined together in any order, their directions remaining unaltered, the end of the last side will still fall upon the beginning of the first.



An example of the construction is shown in the figure, where the sides are changed from the order *abcd* to *adbc*.

4. Express the lengths (x and x') and the directions (θ and θ') of the diagonals of a quadrilateral of which the lengths of the sides, taken in order, are a, b, c, d, and their directions α, β, γ, δ.

It is only necessary to express the values of the quantities $x \sin \theta$, $x \cos \theta$, $x' \sin \theta'$, and $x' \cos \theta'$ in terms of a, b, c, d, α, β, etc. We may suppose x and θ to refer to the diagonal from the beginning of a to the beginning of c, and x' and θ' to run from the end of a to the end of c.

5. Using the same notation as in § 71, prove

$$b \sin (\beta - \alpha) + c \sin (\gamma - \alpha) + d \sin (\delta - \alpha) + \text{etc.} = 0;$$
$$a + b \cos (\beta - \alpha) + c \cos (\gamma - \alpha) + d \cos (\delta - \alpha) + \text{etc.} = 0.$$

6. If α, β, and γ are the angles which the sides a, b, and c of a triangle make with a base line, and A, B, and C are the interior angles of the triangle, it is required—

(1) To show

$$A = 180° + \beta - \gamma; \quad B = 180° + \gamma - \alpha; \quad C = 180° + \alpha - \beta.$$

(2) By combining the equations of Ex. 5, to deduce the law of sines (§ 56, 3) and the fundamental equations (§ 58, 9).

7. From the same point O emanate three lines, OA, OB, OC, of such lengths and directions that the sum of their projections upon any third line vanishes. If we complete the three parallelograms, of each of which two adjacent sides are two of the lines, the areas of these three parallelograms will be equal.

Both a geometric and an algebraic proof may be given; the former from § 70, Th. I., the latter from § 62.

8. From the corners A and B of a pentagonal field $ABCDE$ an engineer measures angles as follows:

Angle $BAC = 79° 23'.6$; Angle $ABC = 47° 29'.7$;
Angle $BAD = 130° 7'.0$; Angle $ABD = 153° 42'.7$;
Angle $BAE = 152° 40'.2$. Angle $ABE = 164° 0'.8$.

AB measures 192 metres. Find the remaining sides and the area.

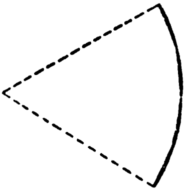
CHAPTER VII.

TRIGONOMETRIC DEVELOPMENTS.[*]

75. Lemma. *When an arc becomes indefinitely small, the ratio of the sine to its arc approaches unity as its limit.*

Remark. In this lemma it is supposed that the arc is expressed in terms of radius as unity.

Proof. It is laid down as an axiom of elementary geometry that when the number of sides of an inscribed regular polygon is indefinitely increased, the perimeter of the polygon approaches the circumference of the circle as its limit. That is, the ratio



$$\frac{\text{perimeter of polygon}}{\text{circumference of circle}} \text{ approaches 1 indefinitely.}$$

Now, if we divide the circumference into n equal arcs, the n chords of these arcs, forming the perimeter of the inscribed polygon, will each be twice the sine of half the arc (§ 18). That is, we shall have

$$\text{Perimeter of polygon} = 2n \times \text{sine of each arc};$$
$$\text{Circumference of circle} = 2n \text{ equal arcs.}$$

Therefore the above ratio is equal to

$$\frac{\text{sine of arc}}{\text{arc itself}},$$

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[*] The study of this chapter requires a knowledge of so much of series and imaginary quantities as is contained in Book XI., Chapters I., II., and V., and Book XII., Chapter I., of the author's "College Algebra." It may be advantageously read in connection with Book XII., Chapter II., of that work. Students taking a partial course may pass to spherical trigonometry without reading this chapter.

which therefore approaches unity as its limit when n is increased indefinitely.

76. Problem. *To develop the sine and cosine of an angle in terms of the ascending powers of the angle.*

Let us suppose

$$\sin x = s_0 + s_1x + s_2x^2 + s_3x^3 + s_4x^4 + s_5x^5 + \text{etc.}; \qquad (1)$$

$$\cos x = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \text{etc.} \qquad (2)$$

in which s_0, s_1, s_2, etc., and c_0, c_1, c_2, etc., are coefficients whose values are to be determined from some known properties of the sine and cosine.

Since sine $(-x) = -\sin x$, the series for $\sin x$ must change its algebraic sign when the algebraic sign of x changes. But only the odd powers of x will then change their sign. Therefore the even powers must not enter into the development, and we must have

$$s_0 = s_2 = s_4 = \text{etc.} = 0.$$

The complete analytic proof of this proposition may be put into the following form. Changing x to $-x$ in the development (1), we have

$$\sin(-x) = s_0 - s_1x + s_2x^2 - s_3x^3 + s_4x^4 - \text{etc.}$$

But we have, by changing the signs of both members of (1),

$$-\sin x = -s_0 - s_1x - s_2x^2 - s_3x^3 - s_4x^4 - \text{etc.}$$

Because these developments must be identically equal, we must have

$$\begin{aligned} s_0 &= -s_0, \\ s_2 &= -s_2, \\ s_4 &= -s_4, \\ \text{etc.} & \quad \text{etc.}, \end{aligned}$$

which gives $s_0 = s_2 = s_4 = \text{etc.} = 0$.

Therefore the development of the sine becomes

$$\sin x = s_1x + s_3x^3 + s_5x^5 + \text{etc.}$$

Dividing this equation by x, we have

$$\frac{\sin x}{x} = s_1 + s_3x^2 + s_5x^4 + \text{etc.}$$

Now suppose x to approach indefinitely near to zero. The first member will then, according to the lemma, approach unity as its limit, and the second member will approach s_1 as its limit. Therefore we must have $s_1 = 1$, and the development becomes

$$\sin x = x + s_3x^3 + s_5x^5 + \text{etc.} \qquad (3)$$

Next take the development (2) for $\cos x$. If we suppose $x = 0$, we have $\cos x = 1$. Therefore, putting $x = 0$, equation (2) will become

$$1 = c_0,$$

which is the required value of c_0.

Again, because $\cos(-x) = \cos x$, the development of $\cos x$ must remain unaltered when we change x into $-x$.

Because this change will reverse the signs of all the odd powers of x, the coefficients of c_1, c_3, etc., of these powers must all vanish, and the development must be

$$\cos x = 1 + c_2 x^2 + c_4 x^4 + c_6 x^6 \ . \ . \ . \qquad (4)$$

77. We must now choose some property of the sine and cosine which will enable us to form equations of condition for the coefficients s_3, s_5 ,etc., and c_2, c_4, c_6, etc. The most simple property for this purpose is that expressed by the addition theorem:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y. \qquad (5)$$

Because the equation (3) is to be true for all values of x, it must remain true when $x + y$ is substituted for x. Making this substitution in (3), we have

$$\sin(x + y) = x + y + s_3(x + y)^3 + s_5(x + y)^5 + \text{etc.}$$

Developing the powers in the second member, and collecting the terms multiplied by the first power of y, we have

$$\left.\begin{aligned} \sin(x + y) = x &+ s_3 x^3 + s_5 x^5 + s_7 x^7 + \ . \ . \ . \\ &+ y(1 + 3s_3 x^2 + 5s_5 x^4 + 7s_7 x^6 + \ . \ . \ .) \\ &+ \text{terms} \times y^2,\ y^3, \text{etc.,} \end{aligned}\right\} \qquad (6)$$

which we need not compute.

From (5) we have, by substituting for $\cos y$ and $\sin y$ their assumed developments, (3) and (4),

$$\left.\begin{aligned} \sin(x + y) &= \sin x(1 + c_2 y^2 + c_4 y^4 + \ . \ . \ .) \\ &\quad + \cos x(y + s_3 y^3 + \text{etc.}) \\ &= \sin x + y \cos x + \text{terms} \times y^2,\ y^3, \text{etc.} \end{aligned}\right\} \qquad (7)$$

Now the expressions (6) and (7) must be identically equal; therefore the coefficients of each power of y must be identically equal. Equating the coefficients of the first power of y, we have

$$1 + 3s_3 x^2 + 5s_5 x^4 + 7s_7 x^6 + \text{etc.} = \cos x.$$

But we have, by (4),

$$\cos x = 1 + c_2 x^2 + c_4 x^4 + \text{etc.}$$

This equation must be satisfied for all values of x. Equating the coefficients of like powers of x, we find

$$\left.\begin{array}{l} 3s_3 = c_2; \\ 5s_5 = c_4; \\ 7s_7 = c_6; \\ \text{etc. etc.} \end{array}\right\} \qquad (8)$$

Next consider the addition theorem for the cosine:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y. \qquad (9)$$

We find, by substituting $x+y$ for x in (4),

$$\cos(x+y) = 1 + c_2(x+y)^2 + c_4(x+y)^4 + \text{etc.};$$

from which, developing to the first power of y as before,

$$\left.\begin{array}{l} \cos(x+y) = 1 + c_2x^2 + c_4x^4 + c_6x^6 + \text{etc.} \\ \qquad + y(2c_2x + 4c_4x^3 + 6c_6x^5 + \text{etc.}) \\ \qquad + \text{terms} \times y^2, y^3, \text{etc.} \end{array}\right\} \qquad (10)$$

From the second member of (9), by substituting for $\sin y$ and $\cos y$ their developments, namely,

$$\cos y = 1 + c^2y^2 + \text{etc.},$$
$$\sin y = y + s_3y^3 + \text{etc.},$$

we find

$$\cos(x+y) = \cos x + y(-\sin x) + \text{terms} \times y^2, y^3, \text{etc.} \qquad (11)$$

Equating the coefficients of y in (10) and (11), we have

$$2c_2x + 4c_4x^3 + 6c_6x^5 + \text{etc.} = -\sin x = -x - s_3x^3 - s_5x^5 - \text{etc.}$$

Equating the coefficients of like powers of x,

$$\left.\begin{array}{l} 2c_2 = -1; \\ 4c_4 = -s_3; \\ 6c_6 = -s_5. \\ \text{etc. etc.} \end{array}\right\} \qquad (12)$$

The equations (8) and (12), taken alternately, solve our problem.

From $(12)_1$, $c_2 = -\dfrac{1}{2}$;

" $(8)_1$, $s_3 = \dfrac{c_2}{3} = -\dfrac{1}{2 . 3}$;

" $(12)_2$, $c_4 = -\dfrac{s_3}{4} = \dfrac{1}{2 . 3 . 4}$;

" $(8)_2$, $s_5 = \dfrac{c_4}{5} = \dfrac{1}{2 . 3 . 4 . 5}$;

From $(12)_6$, $c_6 = -\dfrac{s_5}{6} = -\dfrac{1}{2.3.4.5.6}$;

etc. etc. etc.

The law of the coefficients is obvious. Substituting them in the developments (3) and (4), we have

$$\left.\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \text{etc.};\\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{etc.}\end{aligned}\right\} \qquad (13)$$

78. *Convergence of the series.* These series are convergent for all values of x, a result which may be shown thus:

The ratio of the successive pairs of adjacent terms in the development of the sine are, omitting the minus sign,

$$\frac{x^2}{2.3}, \quad \frac{x^2}{4.5}, \quad \frac{x^2}{6.7}, \quad \frac{x^2}{8.9}, \quad \text{etc.};$$

that is, each term is formed from the preceding one by multiplying by one of these factors. Now, however great may be x, we can continue these fractions so far that their denominators shall become greater than $2x^2$, and so their values less than $\frac{1}{2}$. After this point the sum of all the following terms will be less than that of a geometric progression of which the first term is the term of the development (13), whose quotient by the preceding term is less than $\frac{1}{2}$, and whose ratio is $\frac{1}{2}$. Such a progression has a limit, whence the sum of the series (13) must also have a limit.

The following two applications of these series will be useful as exercises:

1. Square each series, carrying the square as far as the sixth or eighth power of x, and show that the squares fulfil the condition

$$\sin^2 x + \cos^2 x = 1 \textit{ identically}.$$

2. Compute the values of the sine and cosine of 10° and 30° to 5 places of decimals, remembering that we are to take the natural measure of the arc x in radians (§ 14), and compare the result with that in the tables.

We find, from § 14,

$$\text{Arc } 10° = 0.174\,53;$$
$$\text{Arc } 30° = 0.523\,60.$$

79. *Sine and cosine in terms of imaginary exponentials.* It is shown in algebra that if we call e the sum the series,

$$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\text{etc., } \textit{ad infinitum},$$

or

$$e = 2.718\,281\,828\ \ldots\ldots,$$

we shall have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{etc.}$$

Putting i, the imaginary unit, $= \sqrt{-1}$, substituting $xi = x\sqrt{-1}$ for x, and reducing, this equation becomes

$$e^{xi} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \text{etc.} + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \text{etc.}\right)i;$$

or, from the developments (13),

$$e^{xi} = \cos x + i \sin x. \tag{14}$$

Changing xi into $-xi$, we have

$$e^{-xi} = \cos x - i \sin x. \tag{14'}$$

The sum and difference of these equations are

$$\left.\begin{aligned} 2\cos x &= e^{xi} + e^{-xi} = e^{xi} + \frac{1}{e^{xi}};\\ 2i\sin x &= e^{xi} - e^{-xi} = e^{xi} - \frac{1}{e^{xi}},\\ \text{or}\quad 2\sin x &= \frac{1}{i}(e^{xi} - e^{-xi})\\ &= -i(e^{xi} - e^{-xi}). \end{aligned}\right\} \tag{15}$$

For some purposes these equations may be written in the symmetric form

$$\left.\begin{aligned} e^{xi} + e^{-xi} &= \cos x + \cos(-x);\\ e^{xi} - e^{-xi} &= i\sin x - i\sin(-x). \end{aligned}\right\} \tag{15'}$$

These are two of the most celebrated equations in algebraic trigonometry, and are called EULER'S equations, after their discoverer, LEONHARD EULER.

80. *Demoivre's theorem.* If in the equation (14) we substitute nx for x it becomes

$$e^{nxi} = \cos nx + i \sin nx.$$

By raising (14) to the nth power we have

$$e^{nxi} = (\cos x + i \sin x)^n.$$

Therefore $(\cos x + i \sin x)^n = \cos nx + i \sin nx$, (16)
which is known as DEMOIVRE's theorem.

This theorem enables us to develop the sines and cosines of multiples of an angle in powers of the sine or cosine of the simple angle, as follows:

81. PROBLEM. *To develop sin nx and cos nx in powers of sin x and cos x.*

Developing the first member of (16) by the binomial theorem, and substituting for the powers of i their values (Algebra, § 325), namely,

$$\begin{aligned} i^2 &= -1, \\ i^3 &= -i, \\ i^4 &= +1, \\ \text{etc.} &\quad \text{etc.}, \end{aligned}$$

we have*

$$\left.\begin{aligned} (\cos x + i \sin x)^n = \cos^n x &+ \binom{n}{1} i \cos^{n-1} \sin x \\ - \binom{n}{2} \cos^{n-2} x \sin^2 x &- \binom{n}{3} i \cos^{n-3} x \sin^3 x \\ + \binom{n}{4} \cos^{n-4} x \sin^4 x &+ \text{etc.} \end{aligned}\right\} \quad (17)$$

This development being identically equal to the second member of (16), we have, by equating the real terms and putting, for brevity,

$$c = \cos x, \qquad s = \sin x,$$

$$\cos nx = c^n - \binom{n}{2} c^{n-2}s^2 + \binom{n}{4} c^{n-4}s^4 - \binom{n}{6} c^{n-6}s^6 + \text{etc.}$$

This series will go on to infinity unless n is a positive integer,

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* We here use the very convenient abbreviated notation for the binomial coefficients, namely:

$$\binom{n}{1} = \frac{n}{1} = n; \qquad \binom{n}{3} = \frac{n(n-1)(n-2)}{1.2.3};$$

$$\binom{n}{2} = \frac{n(n-1)}{1.2}; \qquad \binom{n}{s} = \frac{n(n-1)(n-2)\ldots(n-s+1)}{1.2.3\ldots s}.$$

We then have $\binom{n}{s} = \binom{n}{n-s}$.

in which case it will terminate with a coefficient $\left(\frac{n}{j}\right)$, in which $j = n$ when n is even, and $j = n - 1$ when n is odd.

If we suppose n equal to 2, 3, 4, etc., in succession, we have

$$\begin{aligned}
\cos 2x &= c^2 - s^2; \\
\cos 3x &= c^3 - 3cs^2; \\
\cos 4x &= c^4 - \frac{4\,.\,3}{1\,.\,2}\,c^2s^2 + s^4; \\
\cos 5x &= c^5 - \frac{5\,.\,4}{1\,.\,2}\,c^3s^2 + \left(\frac{5}{1}\right) cs^4; \\
\cos 6x &= c^6 - \frac{6\,.\,5}{1\,.\,2}\,c^4s^2 + \frac{6\,.\,5}{1\,.\,2}\,c^2s^4 - s^6.
\end{aligned}$$

We may make the results more uniform by substituting for the powers of s their values in powers of c, thus:

$$\left.\begin{aligned}
s^2 &= 1 - c^2; \\
s^4 &= 1 - 2c^2 + c^4; \\
s^6 &= 1 - 3c^2 + 3\,c^4 - c^6; \\
&\text{etc.} \qquad\qquad \text{etc.}
\end{aligned}\right\} \tag{18}$$

Making these substitutions, and reducing the numerical coefficients, we find

$$\left.\begin{aligned}
\cos 2x &= 2c^2 - 1; \\
\cos 3x &= 4c^3 - 3c; \\
\cos 4x &= 8c^4 - 8c^2 + 1; \\
\cos 5x &= 16c^5 - 20c^3 + 5\,c; \\
\cos 6x &= 32c^6 - 48c^4 + 18c^2 - 1;
\end{aligned}\right\} \tag{19}$$

Next equating the coefficients of the imaginary terms of (17) to $i \sin nx$ and dividing by i, we find

$$\begin{aligned}
\sin nx &= \left(\frac{n}{1}\right) c^{n-1} s - \left(\frac{n}{3}\right) c^{n-3} s^3 + \left(\frac{n}{5}\right) c^{n-5} s^5 - \text{etc.} \\
&= s\left\{ \left(\frac{n}{1}\right) c^{n-1} - \left(\frac{n}{3}\right) c^{n-3} s^2 + \left(\frac{n}{5}\right) c^{n-5} s^4 - \text{etc.} \right\}
\end{aligned}$$

Supposing $n = 2, 3, 4, 5$, etc., this form gives

$$\left.\begin{aligned}
\sin 2x &= 2sc; \\
\sin 3x &= s\,\{3c^2 - s^2\}; \\
\sin 4x &= s\,\{4c^3 - 4cs^2\}; \\
\sin 5x &= s\,\{5c^4 - 10\,c^2\,s^2 + s^4\}; \\
\sin 6x &= s\,\{6c^5 - 20c^3\,s^2 + 6cs^4\}; \\
&\text{etc.} \qquad\quad \text{etc.}
\end{aligned}\right\} \tag{20}$$

Substituting for the even powers of s their expressions (18), we find

$$\begin{aligned}\sin 3x &= s\{4c^2 - 1\};\\ \sin 4x &= s\{8c^3 - 4c\};\\ \sin 5x &= s\{16c^4 - 12c^2 + 1\};\\ \sin 6x &= s\{32c^5 - 32c^3 + 6c\}.\end{aligned}$$

Instead of substituting for the powers of s their expressions (18) in terms of the powers of c, we might have expressed the powers of c in terms of s, and by substituting them in (20) have developed the multiple sines in powers of $s = \sin x$.

82. *Expression for powers of the cosine.* The reverse problem, to express the powers of the sine or cosine of an angle in terms of simple sines and cosines of multiples of the angle, is of yet more frequent application.

Let us take the first equation (15),

$$2 \cos x = e^{xi} + e^{-xi},$$

and raise it to the nth power by the binomial theorem. We shall then have

$$2^n \cos^n x = e^{nxi} + \binom{n}{1} e^{(n-2)xi} + \binom{n}{2} e^{(n-4)xi} + \ldots + e^{-nxi}. \quad (21)$$

To understand this general form let the student take the special cases $n = 4$ and $n = 5$. Then

$$2^4 \cos^4 x = e^{4xi} + 4e^{2xi} + \frac{4 \,.\, 3}{1 \,.\, 2} + 4e^{-2xi} + e^{-4xi};$$

$$2^5 \cos^5 x = e^{5xi} + 5e^{3xi} + \frac{5 \,.\, 4}{1 \,.\, 2} e^{xi} + \frac{5 \,.\, 4}{1 \,.\, 2} e^{-xi} + 5e^{-3xi} + e^{-5xi}.$$

Supposing n to be a positive integer, we see (1) that the coefficients of terms equally distant from the two ends of the series are equal, and (2) that the exponents of e in such terms are equal and of opposite signs. Also, if n is even, the middle term does not contain x; but if n is odd, the terms on each side of the middle will contain x.

Therefore by joining the first and last terms, the term after the first and that before the last, etc., the development (21) may be put into the form

$$2^n \cos^n x = e^{nxi} + e^{-nxi} + \left(\frac{n}{1}\right)\{e^{(n-2)xi} + e^{-(n-2)xi}\}$$
$$+ \left(\frac{n}{2}\right)\{e^{(n-4)xi} + e^{-(n-4)xi}\} + \text{etc.}$$

But by the general equations (15′) we have, putting nx for x,

$$e^{nxi} + e^{-nxi} = 2 \cos nx = \cos nx + \cos(-nx),$$

whatever be the value of n.

Hence, substituting this value of the exponential terms,

$$2^n \cos^n x = \left.\begin{array}{l} \cos nx + \left(\frac{n}{1}\right)\cos(n-2)x + \left(\frac{n}{2}\right)\cos(n-4)x + \text{etc.} \\ + \cos(-nx) + \left(\frac{n}{1}\right)\cos(2-n)x + \left(\frac{n}{2}\right)\cos(4-n)x + \text{etc.}, \end{array}\right\} \quad (22)$$

the terms in the third line forming the end of the series, which is doubled over so that the end comes under the beginning.

Giving to n the successive values 2, 3, 4, etc., we find

$$\left.\begin{array}{l} 2^2 \cos^2 x = \cos 2x + 2 + \cos(-2x); \\ 2^3 \cos^3 x = \cos 3x + (\tfrac{3}{1}) \cos x + (\tfrac{3}{2}) \cos(-x) + (\tfrac{3}{3}) \cos(-3x); \\ 2^4 \cos^4 x = \cos 4x + (\tfrac{4}{1}) \cos 2x + (\tfrac{4}{2}) + (\tfrac{4}{3}) \cos(-2x) \\ \qquad\qquad + (\tfrac{4}{4}) \cos(-4x); \\ 2^5 \cos^5 x = \cos 5x + (\tfrac{5}{1}) \cos 3x + (\tfrac{5}{2}) \cos x \\ \qquad + (\tfrac{5}{5}) \cos(-5x) + (\tfrac{5}{4}) \cos(-3x) + (\tfrac{5}{3}) \cos(-x). \end{array}\right\} \quad (23)$$

We have extended the series in this form that the student may see the law of its formation, which is as follows: The successive coefficients are the binomial coefficients. The coefficient of x in the first term is n, and it diminishes by 2 in each following term, so as to become $-n$ in the last term.

The two well-known formulæ

$$\cos(-nx) = \cos nx,$$
$$\left(\frac{n}{s}\right) = \left(\frac{n}{n-s}\right),$$

will enable us to combine every pair of terms equally distant from the extremes into a single one. For instance, we have

$$\left(\frac{6}{4}\right) = \frac{6.5.4.3}{1.2.3.4} = \frac{6.5}{1.2} = \left(\frac{6}{2}\right);$$
$$\left(\frac{5}{4}\right) = \frac{5.4.3.2}{1.2.3.4} = \left(\frac{5}{1}\right) = 5.$$

Combining them thus and dividing each equation by 2, we find

$$\left.\begin{aligned}
2\ \cos^2 x &= \cos 2x + 1;\\
2^2 \cos^3 x &= \cos 3x + 3 \cos x;\\
2^3 \cos^4 x &= \cos 4x + 4 \cos 2x + \frac{1}{2}\cdot\frac{4\,.\,3}{1\,.\,2};\\
2^4 \cos^5 x &= \cos 5x + 5 \cos 3x + \frac{5\,.\,4}{1\,.\,2} \cos x;\\
2^5 \cos^6 x &= \cos 6x + 6 \cos 4x + \frac{6\,.\,5}{1\,.\,2} \cos 2x + \frac{1}{2}\cdot\frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3};\\
&\quad\text{etc.} \qquad\qquad \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}\right\} \quad (24)$$

A careful study of these forms will show the general law of formation, and enable us to carry the powers forward to any extent.

83. *Powers of the sine.* To find the powers of the sine we take the second of equations (15),

$$2i \sin x = e^{xi} - e^{-xi},$$

and raise it to the nth power by the binomial theorem. We then have

$$\left.\begin{aligned}
2^n i^n \sin^n x = e^{nxi} - \left(\frac{n}{1}\right) e^{(n-2)xi} + \left(\frac{n}{2}\right) e^{(n-4)xi} + \text{etc.}\\
\pm e^{-nxi} \mp \left(\frac{n}{1}\right) e^{-(n-2)xi} \pm \left(\frac{n}{2}\right) e^{-(n-4)xi} + \text{etc.}
\end{aligned}\right\} \quad (25)$$

The bottom line gives the last terms of the series, arranged in the reverse order, so that terms equally distant from the extremes are under each other.

The upper signs are to be used in the bottom line when n is *even;* the lower ones when n is *odd.* This will be seen by forming the developments for $n = 5$ and $n = 6$.

$$\left.\begin{aligned}
2^5 i^5 \sin^5 x &= e^{5xi} - \left(\tfrac{5}{1}\right) e^{3xi} + \left(\tfrac{5}{2}\right) e^{xi} - e^{-5xi}\\
&\qquad + \left(\tfrac{5}{4}\right) e^{-3xi} - \left(\tfrac{5}{3}\right) e^{-xi};\\
2^6 i^6 \sin^6 x &= e^{6xi} - \left(\tfrac{6}{1}\right) e^{4xi} + \left(\tfrac{6}{2}\right) e^{2xi} + e^{-6xi}\\
&\qquad - \left(\tfrac{6}{5}\right) e^{-4xi} + \left(\tfrac{6}{4}\right) e^{-2xi} - \left(\tfrac{6}{3}\right).
\end{aligned}\right\} \quad (25')$$

We first take the case of n even. The nth power of i is then $+1$ or -1 according as $\frac{1}{2}n$ is even or odd. Adding the terms of (25) and substituting for the exponential functions their values in terms of cosines from (15'), we find, when n is even,

$$\left.\begin{aligned}(-1)^{\frac{n}{2}}\,2^n \sin^n x = \cos nx - \left(\frac{n}{1}\right)\cos(n-2)x \\ + \left(\frac{n}{2}\right)\cos(n-4)x - \text{etc.}\end{aligned}\right\} \qquad (26)$$

When n is odd, the sum of each pair of corresponding terms in (25) will give rise to a sine. Making the substitution and dividing by i, we shall have

$$\left.\begin{aligned}2^n i^{n-1} \sin^n x = \sin nx - \left(\frac{n}{1}\right)\sin(n-2)x \\ + \left(\frac{n}{2}\right)\sin(n-4)x - \text{etc.}\end{aligned}\right\} \qquad (26')$$

Giving n the successive values 2, 3, 4, etc., and applying the forms (26) and (26′) alternately, and changing all the signs in (26) when $\frac{1}{2}n$ is odd, we find

$$2^2 \sin^2 x = -2\cos 2x + 2.$$

$$2^3 \sin^3 x = -\sin 3x + 3\sin x - 3\sin(-x) + \sin(-3x),$$

or $$2^2 \sin^3 x = -\sin 3x + 3\sin x.$$

$$\begin{aligned}2^4 \sin^4 x = \cos 4x - (\tfrac{4}{1})\cos 2x + (\tfrac{4}{2}) - (\tfrac{4}{3})\cos(-2x) \\ + (\tfrac{4}{4})\cos(-4x),\end{aligned}$$

or $$2^3 \sin^4 x = \cos 4x - 4\cos 2x + 3.$$

$$\begin{aligned}2^5 \sin^5 x = \sin 5x - (\tfrac{5}{1})\sin 3x + \tfrac{5}{2}\sin x - \sin(-5x) \\ + (\tfrac{5}{4})\sin(-3x) - (\tfrac{5}{3})\sin(-x),\end{aligned}$$

or $$2^4 \sin^5 x = \sin 5x - 5\sin 3x + 10\sin x.$$

$$\begin{aligned}2^6 \sin^6 x = -\cos 6x + (\tfrac{6}{1})\cos 4x - (\tfrac{6}{2})\cos 2x + (\tfrac{6}{3}) \\ -\cos(6x) + (\tfrac{6}{5})\cos(-4x) - (\tfrac{6}{4})\cos(-2x),\end{aligned}$$

or $$2^5 \sin^6 x = -\cos 6x + 6\cos 4x - 15\cos 2x + \tfrac{1}{2}\cdot\frac{6.5.4}{1.2.3}.$$

etc. etc.

84. We might have obtained the above results for $\cos x$, $\cos^2 x$, $\cos^3 x$, etc., and $\sin x$, $\sin^2 x$, etc., by consecutive multiplication, and substitution of sines or cosines of sums for products, by § 43. Thus, by § 44, eq. 16′,

$$2\cos^2 x = \cos 2x + 1;$$

× 2 cos x,

$$2^2 \cos^3 x = 2\cos x\cos 2x + 2\cos x.$$

Substituting for the first term of the second member its values (12) of § 43, we have

$$2^2 \cos^3 x = \cos 3x + \cos x + 2 \cos x = \cos 3x + 3 \cos x.$$

Multiplying this equation by $2 \cos x$, we should have an expression for $\cos^4 x$, etc. But the use of exponentials enables us not only to obtain the higher powers more expeditiously, but to find the general law of the series, which is not readily done by multiplication.

EXERCISES.

1. In the expression

$$1 + 2 \cos x + 3 \cos^2 x + 4 \cos^3 x,$$

substitute for the powers of x their values in terms of the multiple of x, and reduce the expression to one containing simple multiples of x.

Solution. From (24),

$$\begin{array}{llllll}
4 \cos^3 x & = \cos 3x & & & + 3 \cos x & \\
3 \cos^2 x & = & & + \frac{3}{2} \cos 2x & & + \frac{3}{2} \\
2 \cos x & = & & & + 2 \cos x & \\
1 & = & & & & 1 \\
\hline
 & \text{Sum} = \cos 3x & & + \frac{3}{2} \cos 2x & + 5 \cos x & + \frac{5}{2}
\end{array}$$

2. Reduce the expressions

$$\begin{aligned}
&4 \cos^3 x - 3 \cos x, \\
&8 \cos^4 x - 8 \cos^2 x + 1, \\
&16 \cos^5 x - 20 \cos^3 x + 5 \cos x, \\
&32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1,
\end{aligned}$$

to terms containing sines and cosines of multiples of x, thus proving eq. (19).

3. Prove that the expression

$$1 - 2\alpha \cos \theta + \alpha^2$$

may be resolved into the two factors $(1 - \alpha e^{\theta i})\,(1 - \alpha e^{-\theta i})$.

4. Resolve the expression $x^{2n} - 2x^n \cos \theta + 1$ into the product of two factors, as in the last example.

Trigonometric Forms of Imaginary Expressions.

85. It is shown in algebra that an imaginary or complex expression may be reduced to a certain number of real units *plus* a certain number of imaginary units. If we put

i, the imaginary unit, $= \sqrt{-1}$,
a, the number of real units,
b, the number of imaginary units,

the complex expression will be

$$a + bi. \tag{1}$$

We have already shown (§ 47) that, whatever be the numbers a and b, we can find a positive number r and an angle φ, such that

$$r \cos \varphi = a;$$
$$r \sin \varphi = b.$$

If we substitute these values of r and φ in (1) it will become

$$a + bi = r(\cos \varphi + i \sin \varphi).$$

But equation (14) gives

$$\cos \varphi + i \sin \varphi = e^{\varphi i}.$$

Therefore
$$a + bi = re^{\varphi i}. \tag{2}$$

We hence conclude:

Every complex expression can be reduced to the form

$$re^{\varphi i},$$

which is called the general form of the complex expression.

The coefficient r is called the **modulus** of the expression.

A yet better term, used by the Germans, is the "absolute value" of the expression.

The angle φ is called the **argument** of the expression.

Example. Reduce the expression

$$-0.9223 + 1.0962i$$

to the general form.

Putting
$$r \cos \varphi = -0.9223,$$
$$r \sin \varphi = 1.0962,$$

and applying the process of § 47, we find

$$r = 1.4326;$$
$$\varphi = 130° \; 4'.54.$$

This process being purely algebraic, the angle φ should be expressed in radial units. Reducing to this unit, we find

$$\varphi = 2.2703.$$

Therefore the required general form is

$$-0.9223 + 1.0962i = 1.4326e^{2.2703i}.$$

The student who is acquainted with the geometric representation of imaginary quantities will see that the quantity r corresponds to the modulus and φ to the angle of the complex expression as defined in algebra.

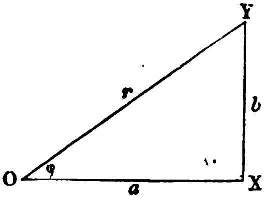
The geometric construction of the expression $a + bi$ is effected by laying off the length a on the axis of X, and at the end of this length erecting a perpendicular equal to b.

If O be the origin, we shall have

$$OX = a;$$
$$XY = b.$$

Then joining OY we shall have

$$OY = r;$$
$$\text{Angle } XOY = \varphi.$$



86. *Multiplication of complex expressions in the general form.* If any two complex expressions are

$$re^{\phi i} \text{ and } qe^{\theta i},$$

we have by multiplying them

$$rqe^{(\phi+\theta)i}.$$

This is another complex expression of the general form of which rq is the modulus and $\varphi + \theta$ the argument. Hence:

The modulus of a product is equal to the product of the moduli of the factors.

The argument of a product is the sum of the arguments of the factors.

If we multiply n equal factors, each represented by $re^{\phi i}$, the result will be

$$(re^{\phi i})^n = r^n e^{n\phi i}.$$

Hence:

The modulus of a power is equal to the corresponding power of the modulus of the root.

The argument of the power is the argument of the root multiplied by the index of the power.

87. *Periodicity of the imaginary exponential.* From the known equations (§ 24)

$$\cos(\varphi + 2\pi) = \cos\varphi,$$
$$\sin(\varphi + 2\pi) = \sin\varphi,$$

and the following equations given by the preceding theory,

$$e^{(\phi + 2\pi)i} = \cos(\varphi + 2\pi) + i\sin(\varphi + 2\pi),$$
$$e^{\phi i} = \cos\varphi + i\sin\varphi,$$

we have

$$re^{(\phi + 2\pi)i} = re^{\phi i};$$

that is:

The value of a complex quantity remains unaltered when we increase its argument by a circumference.

Since the addition of one circumference does not change it, the addition of any number of circumferences will still leave it unchanged. Hence:

If the argument of a complex quantity increases indefinitely, the values of the quantity itself will repeat themselves with every circumference by which the argument increases.

A quantity whose value repeats itself in this way is said to be **periodic.**

88. Let us next inquire for what special values of φ the exponential function $e^{\phi i}$ will be equal to the real or imaginary unit. Considering again the equation

$$e^{\phi i} = \cos\varphi + i\sin\varphi,$$

we notice that $\sin\varphi = 0$ whenever φ is a multiple of 180° or of π. When the multiple of π is even, we have $\cos\varphi = +1$; and when it is odd, $\cos\varphi = -1$. Hence, putting

$$\varphi = \pi,\ 2\pi,\ 3\pi,\ \text{etc.},$$

we have

$$\left.\begin{array}{ll} e^{\pi i} = -1; & e^{-\pi i} = -1; \\ e^{2\pi i} = +1; & e^{-2\pi i} = +1; \\ e^{3\pi i} = -1; & e^{-3\pi i} = -1; \\ \text{etc.} & \text{etc.} \end{array}\right\} \qquad (a)$$

In order that $\cos\varphi$ may vanish, the angle φ must be 90°, 270°, 450°, etc.; that is, it must be an odd multiple of $\frac{1}{2}\pi$. Sin φ will then be $+1$ or -1. Putting $\varphi = \frac{1}{2}\pi$, $\varphi = \frac{3}{2}\pi$, $\varphi = \frac{5}{2}\pi$, etc., on both sides of the preceding equation, we have

$$\left.\begin{array}{ll} e^{\frac{1}{2}\pi i} = +i; & e^{-\frac{1}{2}\pi i} = -i; \\ e^{\frac{3}{2}\pi i} = -i; & e^{-\frac{3}{2}\pi i} = +i; \\ e^{\frac{5}{2}\pi i} = +i; & e^{-\frac{5}{2}\pi i} = -i. \end{array}\right\} \qquad (b)$$

By squaring each of these equations we shall reproduce the alternate equations (*a*).

89. *Roots of unity.* The foregoing theory enables us to find very simple and elegant expressions for the roots of the equation

$$x^n - 1 = 0,$$

or

$$x^n = 1.$$

From the general theory of equations, the equation $x^n - 1 = 0$, being of the nth degree, must have n roots; that is, there are n quantities which, being raised to the nth power, will produce 1.

These quantities are called the nth *roots of unity.*

Because 1^n is always 1, whatever be n, $+1$ is itself one of the nth roots of unity.

Because $(-1)^n = 1$ when n is even, -1 is always an nth root of unity when n is even.

Hence one or two of the n roots of unity, viz. $+1$ and -1, are real; all the others are imaginary.

90. PROBLEM. *To find the nth roots of unity.*

Solution. Let a required root be $re^{\theta i}$, r and θ being quantities to be determined. By the requirements of the problem, the nth power of this quantity must be 1. Its nth power is

$$(re^{\theta i})^n = r^n e^{n\theta i} = r^n (\cos n\theta + i \sin n\theta).$$

In order that this expression may be equal to unity, a real quantity, the coefficient of i must vanish, and we must have

$$\sin n\theta = 0,$$

which gives

$$\cos n\theta = 1.$$

Hence

$$r^n = 1,$$

which is satisfied by supposing

$$r = 1.$$

We must also have

$$n\theta = 0 \text{ or } 2\pi \text{ or } 4\pi \text{ or } 6\pi, \text{ etc.}$$

Dividing by n, we see that θ may have any one of the values

$$\theta = 0,$$

$$\theta = 2\frac{\pi}{n},$$

$$\theta = 4\frac{\pi}{n},$$

$$\theta = 6\frac{\pi}{n},$$

etc.

By substituting, in the assumed expression, $re^{\theta i}$ for the value of the root, we have

$$n\text{th roots of } 1 = 1, e^{\frac{2\pi}{n}}, e^{\frac{4\pi}{n}}, e^{\frac{6\pi}{n}}, \text{etc.}$$

Reducing to the trigonometric form, these expressions become

$$1;$$

$$\cos 2\frac{\pi}{n} + i \sin 2\frac{\pi}{n};$$

$$\cos 4\frac{\pi}{n} + i \sin 4\frac{\pi}{n};$$

$$\cos 6\frac{\pi}{n} + i \sin 6\frac{\pi}{n};$$

etc. etc.

The angle increases by $2\frac{\pi}{n}$ with each root, and by writing n consecutive values we shall be carried all round the circle.

The solution which we thus reach may be represented thus:

Divide the circle into n equal arcs.

Let the length of each arc be α, *so that* $n\alpha = 360° = 2\pi$.

The nth roots of unity will be:

$$\left.\begin{array}{l} \cos 0 \ + i \sin 0 = 1; \\ \cos \alpha \ + i \sin \alpha; \\ \cos 2\alpha + i \sin 2\alpha; \\ \cos 3\alpha + i \sin 3\alpha; \\ \quad . \quad . \quad . \quad . \quad . \quad . \\ \cos (n-1)\alpha + i \sin (n-1)\alpha. \end{array}\right\} \qquad (3)$$

Example. To find the sixth roots of 1. Here

$$n = 6; \qquad \frac{\pi}{n} = \frac{180°}{n} = 30°; \qquad 2\frac{\pi}{n} = 60° = \alpha.$$

Hence the six roots are

$$\left.\begin{aligned}&1;\\ &\cos 60^\circ + i\sin 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2}i;\\ &\cos 120^\circ + i\sin 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i;\\ &\cos 180^\circ + i\sin 180^\circ = -1;\\ &\cos 240^\circ + i\sin 240^\circ = -\frac{1}{2} - \frac{\sqrt{3}}{2}i;\\ &\cos 300^\circ + i\sin 300^\circ = \frac{1}{2} - \frac{\sqrt{3}}{2}i.\end{aligned}\right\} \qquad (4)$$

The result can be readily proved by raising each of these quantities to the sixth power.

The roots may also be constructed as in the anexed figure.

If angle $XOA = AOB =$ etc. $= 60^\circ$, then, since $\frac{PA}{OX} = \sin 60^\circ$ and $\frac{OP}{OX} = \cos 60^\circ$,

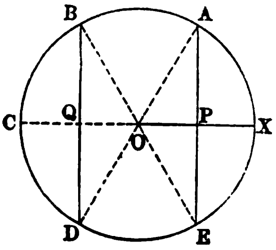
$$\frac{OX}{OX} = 1,$$

$$\frac{OP}{OX} + \frac{PA}{OX}i,$$

$$\frac{OQ}{OX} + \frac{BQ}{OX}i,$$

etc. etc.,

are the sixth roots of unity.



EXERCISES.

1. Find and construct the eighth roots of unity, or the roots of the equation $x^8 - 1 = 0$.

2. Find the roots of the equation $x^{12} - 1 = 0$.

91. *Relations between the roots of unity.* If we represent by x any such quantity as $\cos\alpha + i\sin\alpha$, we have, by what precedes,

$$x^2 = \cos 2\alpha + i\sin 2\alpha;$$
$$x^3 = \cos 3\alpha + i\sin 3\alpha;$$
$$x^n = \cos n\alpha + i\sin n\alpha.$$

Hence the formation of the powers of x may be represented geometrically by laying off equal arcs around a circle.

If x is any nth root of unity, then measuring off its angle α n times will bring us back to the starting-point.

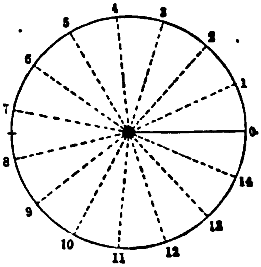
If α is itself the nth part of the circumference, then the remaining roots as given in (3) are the first n powers of x. Hence:

All the roots of unity are powers of the root corresponding to the smallest arc.

From this it follows that if we measure off with a pair of dividers, from 0 to any division-point, the mth, for instance, and repeat the measure n times, the nth measure will end at the zero-point.

This is evident of itself, because n measures of m arcs each will measure off mn arcs; and because n of the arcs make up a circumference, the mn arcs will extend around the circle exactly m times.

But it does not follow that any such series of n measures will include all the roots. Suppose, for example, that in the preceding figure, where $n = 15$, we measure arcs of 6α. The 15 successive points reached with the dividers will then be 0, 6, 12, 3, 9, 0, 6, 12, 3, 9, 0, 6, 12, 3, 9, 0. This series includes only 5 of the points of division, each of these 5 being repeated 3 times, while the remaining 10 have not been included at all.



If we take the measure 4α in our dividers, the points of division included in the series will be

0, 4, 8, 12, 1, 5, 9, 13, 2, 6, 10, 14, 3, 7, 11, 0,

which comprise all n points. Hence in this case all the roots are powers of x^4 or of $\cos 4\alpha + i \sin 4\alpha$.

92. We now have the following proposition, where we put

$$\alpha = \frac{360^\circ}{n}:$$

If m is prime to n, all the nth roots of unity may be represented as powers of $\cos m\alpha + i \sin m\alpha$.

Proof. Starting from any point of the circle, and measuring off equal arcs, each of length $m\alpha$, let p be the smallest number of measures which will bring us back to the starting-point. The total length of arc measured off will then be $pm\alpha$.

Since we are brought back to the starting-point, we must have measured off an entire number of circumferences. Let q be that entire number.

Because each circumference $= n\alpha$, the whole q circumferences measured $= qn\alpha$. Therefore

$$pm\alpha = qn\alpha,$$

and

$$\frac{p}{q} = \frac{n}{m}.$$

Because, by hypothesis, m and n are prime to each other, the fraction $\frac{n}{m}$ is irreducible, and the smallest values of p and q are n and m respectively.

Therefore any n measures will end at n *different* points of the circumference, and will therefore include all n points.

Def. A root of unity whose powers include all other roots is called a *primitive root.*

Cor. If n is a prime number, all the roots are primitive roots.

EXERCISES.

1. The 15th roots of unity being

$$\cos 24° + i \sin 24°, \qquad \cos 48° + i \sin 48°, \qquad \text{etc.},$$

it is required to find which of these roots are primitive.

Prove the following propositions:

2. If n is a prime number, all the nth roots of unity are primitive roots.

3. If x be any primitive nth root of unity, and if p be any number prime to n, then x^p will also be a primitive root.

NOTE. In the preceding theorem this is proved for the case when x is the root corresponding to the smallest angle. The proposition now enunciated extends to the case in which we start from any multiple of this angle prime to n.

PART II.

SPHERICAL TRIGONOMETRY.

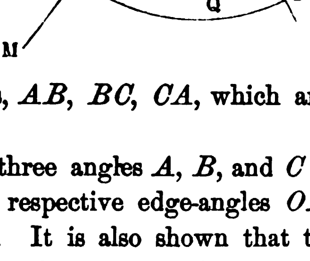
CHAPTER I.

FUNDAMENTAL PRINCIPLES.

93. *Def.* Spherical trigonometry treats of the relations among the six parts of a trihedral angle.

Def. The six parts of a trihedral angle are its three face-angles and its three edge-angles.

94. *Representation of a trihedral angle by a spherical triangle.* If O be the vertex of a trihedral angle, and OM, OP, and OQ its three edges, we may construct a sphere having its centre at O, and having an arbitrary radius OA. The spherical surface will then cut the edges at the three points A, B, and C equally distant from O.



The three faces OMP, OPQ, OQM will intersect the spherical surface in three arcs of great circles, AB, BC, CA, which arcs form a spherical triangle.

It is shown in geometry that the three angles A, B, and C of the spherical triangle are equal to the respective edge-angles OM, OP, and OQ of the trihedral angle. It is also shown that the arcs AB, BC, and CA, which form the sides of the triangle, measure the respective face-angles MOP, POQ, QOM of the trihedral angle.

Therefore the six parts of the trihedral angle are represented by the corresponding parts of the spherical triangle, and the relations among the parts of the one are the same as the relations among the parts of the other.

The term *spherical trigonometry* is applied because the investigations are generally made by means of the spherical triangle.

A trihedral angle, with its corresponding spherical triangle, may be readily constructed as follows: Cut a circular disk of pasteboard or stiff paper, from four to six inches or more in diameter. From this disk cut out a sector of any magnitude. It will be well to have several disks with sectors ranging from 45° to 200° cut out. Divide the remainder of the disk by two radii into three sectors, such that the greatest shall be less than the sum of the other two. Bend the disk along each of the two dividing radii, cutting the latter part of the way through if necessary, and bring the extreme radii together. We shall then have a figure like O-ABC of the preceding diagram, the three plane sides forming the trihedral angle, and the three arcs bounding the edge of the disk forming the spherical triangle.

95. *General remarks upon spherical triangles.* A spherical triangle may be defined as that figure which is formed by joining any three points on the surface of a sphere by arcs of great circles. The three points will then be the vertices of the triangle.

But between any two points we may draw two arcs of a great circle, which together make up a complete great circle through the points. One of these arcs will be less, the other greater, than 180°. To avoid ambiguity, the arc less than 180° is supposed to be taken, unless otherwise expressed. We therefore adopt the rule:

Each side of a spherical triangle is supposed less than 180°, *unless otherwise expressed.*

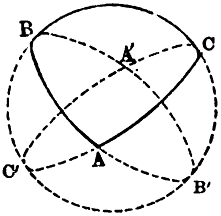
This rule is a mere convention, which may be set aside whenever we desire to give greater generality to our conclusions. Nothing prevents us from supposing ourselves to pass from one vertex to another by passing several times around the sphere. The corresponding side of the triangle will then consist of several coincident great circles *plus* either of the arcs joining the vertices. If we suppose a to be the shorter arc joining two vertices, the general arc measure of the side through those vertices will be

$$n\,360^\circ + a \qquad \text{or} \qquad (n + 1)\,360^\circ - a.$$

96. Every spherical triangle encloses a portion of the spherical surface, forming the area of the triangle. We then have the theorem:

Three great circles divide the surface of the sphere into eight triangular portions.

This is shown as follows: One great circle divides the surface into two equal parts. A second great circle intersects the first in two points, and divides each of those parts into two lunes, so that the whole surface is then divided into four lunes. A third great circle cuts through all four of these lunes, and forms eight spherical triangles.



The eight spherical triangles formed by three great circles.

In the same way, any two planes divide the space around their line of intersection into four parts. A third plane intersecting them divides the space around their point of intersection into eight parts, forming eight trihedral angles.

Remark. The student should guard himself against considering a figure of which either side is a small circle of the sphere as a spherical triangle. For example, the figure formed by two arcs of meridians and a parallel of latitude is not a spherical triangle. Such figures do not represent the parts of a trihedral angle, and so do not correspond to the definition of a spherical triangle. All the important problems connected with them may be reduced to problems of spherical trigonometry, so that there is no need of giving them special consideration.

EXERCISES.

The following exercises are introduced to test the student's fundamental conceptions of spherical geometry, and especially of the relations of great circles of the sphere. Their successful performance will show that he is prepared to take up the subject of spherical trigonometry with advantage. A globe, on which figures may be drawn at pleasure, will be of great service in assisting his conceptions, and should be made use of whenever practicable.

1. A and A' are two opposite points on a sphere. If any third point X be taken on the sphere, to what constant arc will the sum $XA + XA'$ be equal, and what will be the angle AXA'?

NOTE. Opposite points are those at the ends of a diameter.

2. If one side of a spherical triangle be equal to a semicircle, what relations will then subsist between the other two sides? What will be the magnitude of the opposite angle?

3. Let A, B, and C be the three vertices of a spherical triangle; a, b, and c, the sides opposite these vertices respectively; A', B', and C', the points opposite the vertices. It is then required:

(a) To express by the letters at their vertices the eight triangles which will be formed when each side of the original triangle ABC is produced into a great circle.

(b) To express the sides of each of these eight triangles in terms of a, b, and c, making use of the theorem that any two great circles intersect each other in two opposite points.

(c) To express the angles in terms of the angles of the original triangle, which we may represent by the letters A, B, and C marking their vertices.

(d) It being found that the eight triangles are divisible into four pairs, such that the sides and angles of each pair are equal, it is required to show the relations of each pair.

4. If one angle of a spherical triangle is A, show that the sum of the other two angles is contained between the limits $180° - A$ and $180° + A$.

NOTE. If the student finds any difficulty in this question he may begin by supposing the triangle to be isosceles, and the two equal sides to increase from 0° to 180°.

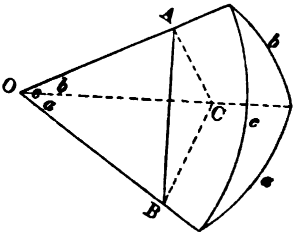
5. Hence show that the spherical excess cannot exceed twice the smallest angle.

6. If the three sides of an equilateral spherical triangle be continually and equally increased, what is the limit of their sum? What is the limit of the angles as the sum approaches its limit?

97. *Fundamental equations.* Let us put

a, b, c, the three face-angles of the trihedral angle—that is, the angles subtended by the three sides of the spherical triangle;

A, B, C, the opposite edge-angles of the trihedral angle, or the angles of the spherical triangle.



Then if $O\text{-}ABC$ be any trihedral angle, we shall have

$$a = \text{angle } BOC;$$
$$b = \text{angle } COA;$$
$$c = \text{angle } AOB.$$

Through any point A of OA pass a plane perpendicular to OA, and let B and C be the points in which it meets the other two edges. We shall then have

$$A = \text{angle } BAC,$$

while OAB and OAC will both be right angles.

In the triangle BOC we have

$$BC^2 = OB^2 + OC^2 - 2OB \, . \, OC \cos a. \qquad (\S\ 58)$$

In the triangle BAC we have

$$BC^2 = AB^2 + AC^2 - 2AB \, . \, AC \cos A.$$

Equating these two values of BC^2, and transposing, we find

$$2OB \, . \, OC \cos a = OB^2 - AB^2 + OC^2 - AC^2 + 2AB \, . \, AC \cos A.$$

But, because OAB and OAC are right angles,

$$OB^2 - AB^2 = OA^2;$$

$$OC^2 - AC^2 = OA^2.$$

Substituting these values, and dividing by $2OB \, . \, OC$, we have

$$\cos a = \frac{OA}{OB} \cdot \frac{OA}{OC} + \frac{AB}{OB} \cdot \frac{AC}{OC} \cos A.$$

Now

$$\frac{OA}{OB} = \cos c;$$

$$\frac{OA}{OC} = \cos b;$$

$$\frac{AB}{OB} = \sin c \cdot$$

$$\frac{AC}{OC} = \sin b.$$

Therefore $\quad \cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad (a)$

By treating the other edges in order in the same way, we obtain two more equations, which may be written by simply permuting a, b, and c and A, B, and C *circularly*; that is, by substituting for each letter the one next in order, a following c. Thus we have the system of three equations:

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A; \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B; \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C. \end{aligned}\right\} \qquad (1)$$

These three equations are the fundamental equations of spherical trigonometry, because by means of them, when three parts are given, the other three may be found. For practical application they are transformed and simplified in numerous ways.

98. *Permutation of parts.* We have deduced the equations (1) from (*a*) by merely permuting the letters. This process may be applied generally in accordance with the following theorem:

If we have found between the parts of a spherical triangle any equation which is true for all triangles, it will remain true when we permute the sides in any way; provided that we also permute the opposite angles in the same way.

For if we have proved our equation by calling the three sides a, b, and c, and the opposite angles A, B, and C, we could apply the same proof to the other parts of the triangle, substituting

Side a for side b, and *vice versa*,

and

Angle A for angle B, and *vice versa*,

in the demonstration. We should then have a result in which a and b changed places, and A and B changed places.

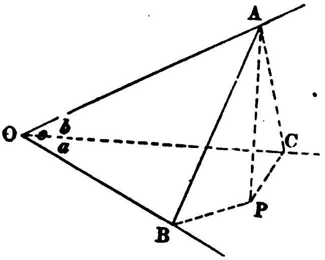
By interchanging a and c, and b and c, with their opposite angles in the same way, we should form all the six equations which could be written by permuting the symbols in the way described.

If, however, we made any supposition respecting any side or angle such that the reasoning applied to it would not apply to the others, then the symbol of this side or angle could not be permuted. For instance, we cannot permute all the parts in a formula true only for right triangles.

It follows from this that any true formula which expresses the value of one part in terms of the two remaining pairs of parts must be symmetrical with respect to the other pairs of parts. For example, equation a remains unchanged when we interchange b and c, else it would be wrong.

99. Theorem of sines. *In a spherical triangle the sines of the sides are proportional to the sines of the opposite angles.*

Proof. Let $O\text{-}ABC$ be the trihedral angle of the spherical tritriangle, and let A be any point on the edge OA.



Through A pass a plane perpendicular to the edge OB, intersecting the faces AOB and BOC in the lines AB and BP.

Through A pass another plane perpendicular to OC, intersecting the faces AOC and COB in the lines AC and CP.

AP will then be the line of intersection of these planes.

Because the planes ABP and ACP are perpendicular to the lines OB and OC respectively, they are each perpendicular to the plane BOC of these lines (*Geom.*) Therefore their line of intersection, AP, is also perpendicular to this plane, and the triangles APB and APC are right-angled at P. Hence

$$AP = AB \sin ABP = AB \sin B.$$

Also, because AB is perpendicular to OB,

$$AB = OA \sin BOA = OA \sin c.$$

Therefore $AP = OA \sin c \sin B$.

We find in the same way

$$AP = OA \sin b \sin C;$$

whence

$$\sin c \sin B = \sin b \sin C,$$

or

$$\frac{\sin c}{\sin C} = \frac{\sin b}{\sin B}.$$

We may show in the same way, by permuting the parts,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \qquad (2)$$

The common value of these three ratios is called the *modulus* of the spherical triangle.

100. The theorem of sines may also be obtained directly from the fundamental equations as follows:

From the first fundamental equation (1) we obtain

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Squaring,

$$\cos^2 A = \frac{\cos^2 a - 2\cos a \cos b \cos c + \cos^2 b \cos^2 c}{\sin^2 b \sin^2 c}$$

Hence

$$\sin^2 A = 1 - \cos^2 A =$$

$$\frac{(1 - \cos^2 b)(1 - \cos^2 c) - \cos^2 a + 2\cos a \cos b \cos c - \cos^2 b \cos^2 c}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Dividing by $\sin^2 a$,

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}.$$

The second member of this equation is symmetrical with respect to a, b, and c, and so remains unchanged when these quantities are permuted among themselves. But if we derive the values of $\frac{\sin^2 B}{\sin^2 b}$ and $\frac{\sin^2 C}{\sin^2 c}$ from the last two fundamental equations, the results will be simple permutations of the last equation, and will therefore give the same values of $\frac{\sin^2 B}{\sin^2 b}$ and $\frac{\sin^2 C}{\sin^2 c}$ that we have found for $\frac{\sin^2 A}{\sin^2 a}$.

Hence

$$\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c}.$$

Extracting the square roots, the general results will have double (±) algebraic signs; but as the angles are all supposed to be less than 180°, the positive signs are to be taken. Hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

the reciprocal of the relations (2).

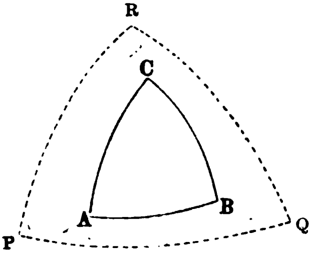
101. *Polar triangles. Def.* When two triangles are so related that the vertices of the one are the poles of the sides of the other, the one is said to be the **polar triangle** of the other.

It is shown in geometry that the relation of a triangle to its polar triangle is reciprocal; that is, if X and Y are two triangles, and Y is the polar triangle of X, then X is the polar triangle of Y. This reciprocity arises from the theorem:

If A, B, and C be the three poles of the sides QR, RP, and PQ of a triangle PQR, then P, Q, and R will be the poles of the sides BC, CA, and AB.

This theorem is readily proved by the geometry of the sphere.

Since every great circle has two poles, one at each end of a diameter, it follows that the three sides of a triangle have six poles in all. We may form a polar triangle to ABC by taking either of the poles of AB, either of the poles of BC, and either of the poles of CA, and joining them by arcs of great circles. Hence there are eight possible polar triangles to every given triangle. To avoid doubt which triangle is to be chosen, we take for each vertex of the polar triangle that pole of each side of the given triangle which is on the side toward the triangle.



For example, if ABC is the given triangle, we take that pole of AB which is on the side toward C, and so with the other sides.

EXERCISES.

1. What must be the sides and angles of a triangle that it may coincide with its polar triangle?

2. Show that if each side of a triangle is greater than 90° the polar triangle will fall wholly inside of it, and if each side is less than 90° it will be wholly within its polar triangle.

3. If two sides exceed 90° and the third side is less than 90°, what will be the character of the polar triangle, and how will it be situated relatively to the given one?

102. *Use of the polar triangle.* It is shown in geometry that each side of the polar triangle is the supplement of the opposite angle of the other, and *vice versa.* This principle is applied to find new relations between the parts of a triangle in the following way:

1. We imagine ourselves to construct the polar of the given triangle.

2. We write any or all the equations between the parts of the polar triangle.

3. We substitute in these equations the supplementary parts of the given triangle, and thus obtain equations between these parts.

Let us put
$$a',\ b',\ c',$$
$$A',\ B',\ C',$$
the sides and opposite angles of the polar triangle. Since the general equations (1) are true for every triangle, they are true of this polar triangle. Hence
$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$
But the polar triangle is so related to the original triangle that
$$\begin{aligned} a' &= 180° - A, & A' &= 180° - a; \\ b' &= 180° - B, & B' &= 180° - b; \\ c' &= 180° - C, & C' &= 180° - c. \end{aligned}$$
Therefore
$$\begin{aligned} \cos a' &= -\cos A, & \cos A' &= -\cos a; \\ \cos b' &= -\cos B, & \cos B' &= -\cos b; \\ \cos c' &= -\cos C, & \cos C' &= -\cos c; \end{aligned}$$
and
$$\sin a' = \sin A, \qquad \sin A' = \sin a;$$
$$\text{etc.} \qquad\qquad \text{etc.}$$
Making these substitutions in the equations (1), we find
$$\left.\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a; \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b; \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c. \end{aligned}\right\} \quad (3)$$
This process may be generalized thus:

From every relation between the parts of a spherical triangle we may derive another relation by changing the cosine of each part into the negative of the cosine of the opposite part, and the sine of each part into the sine of the opposite part.

But this relation will not always be different from the original one. If we apply the process to the equations (2), for instance, the same relations will be reproduced, each term being changed to its reciprocal.

It is also to be remarked that the use of the polar triangle is not absolutely necessary to deduce the new relations (3), because they can all be obtained from the fundamental equations (1) by eliminating first b and c, then c and a, then a and b. But the use of the polar triangle gives a much shorter and more elegant mode of deducing them.

103. In the solution of spherical triangles we require formulæ which shall express any three parts in terms of the remaining three; the latter being supposed known, the former unknown.

A set of such equations may be derived from the fundamental equations (1) by substituting in any one the value of the cosine of a side obtained from another. Let us substitute in the third the value of cos a from the first. We shall have

$$\cos c = \cos^2 b \cos c + \sin b \cos b \sin c \cos A + \sin a \sin b \cos C.$$

Transposing cos c, noting that $1 - \cos^2 b = \sin^2 b$, and dividing by sin b, we have

$$0 = -\sin b \cos c + \cos b \sin c \cos A + \sin a \cos C,$$

which gives

$$\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A. \qquad (4)$$

The theorem of sines (2) also gives

$$\sin a \sin C = \sin c \sin A. \qquad (5)$$

Comparing these two last equations with the first equation (1), we see that they form a set in which the second members contain only the parts b, c, and A, namely, two sides and the included angle; while the first members contain the third side, a, and one of the angles adjacent to it, namely, C.

Hence any two of these three equations may be used to find the side a and the angle C when b, c, and A are given. Since there are three equations where only two are necessary, there must be a relation between them, which we find as follows:

The first members are

$$\cos a;$$
$$\sin a \cos C;$$
$$\sin a \sin C.$$

The sum of the squares of these quantities is

$$\cos^2 a + \sin^2 a(\cos^2 C + \sin^2 C) = \cos^2 a + \sin^2 a = 1.$$

The sum of the squares of the first members being identically equal to unity, the same should be true of the sum of the squares of the second members, which is

$$\begin{aligned}&(\cos b\cos c+\sin b\sin c\cos A)^2\\+&(\sin b\cos c-\cos b\sin c\cos A)^2\\+&\sin^2 c\sin^2 A.\end{aligned}$$

Forming these squares, we see that the product of the terms in the squares of the first two quantities cancel each other, leaving as the sum of the squares

$$\begin{aligned}&\cos^2 b\cos^2 c+\sin^2 b\sin^2 c\cos^2 A\\+&\sin^2 b\cos^2 c+\cos^2 b\sin^2 c\cos^2 A\\+&\sin^2 c\sin^2 A,\end{aligned}$$

a sum which we readily find to reduce to unity.

104. By permuting the sides b and c and their opposite angles, B and C, we obtain a similar set of equations for a and the adjacent angle B. Repeating the equations already found, we have the following set of five equations from which we may determine a, B, and C, when b, c, and A are given:

$$\left.\begin{aligned}\sin a\sin B&=\sin b\sin A;\\\sin a\cos B&=\cos b\sin c-\sin b\cos c\cos A;\\\sin a\sin C&=\sin c\sin A;\\\sin a\cos C&=\sin b\cos c-\cos b\sin c\cos A;\\\cos a&=\cos b\cos c+\sin b\sin c\cos A.\end{aligned}\right\}\qquad(6)$$

We may write a set similar to this for each of the other sides of the triangle, namely:

$$\left.\begin{aligned}\sin b\sin C&=\sin c\sin B;\\\sin b\cos C&=\cos c\sin a-\sin c\cos a\cos B;\\\sin b\sin A&=\sin a\sin B;\\\sin b\cos A&=\sin c\cos a-\cos c\sin a\cos B;\\\cos b&=\cos c\cos a+\sin c\sin a\cos B.\end{aligned}\right\}\qquad(7)$$

$$\left.\begin{aligned}\sin c\sin A&=\sin a\sin C;\\\sin c\cos A&=\cos a\sin b-\sin a\cos b\cos C;\\\sin c\sin B&=\sin b\sin C;\\\sin c\cos B&=\sin a\cos b-\cos a\sin b\cos C;\\\cos c&=\cos a\cos b+\sin a\sin b\cos C.\end{aligned}\right\}\qquad(8)$$

Then we obtain a similar set for the angles through the intervention of the polar triangle. Applying the set (6) to the polar triangle, it gives

$$\left.\begin{aligned}
\sin A \sin b &= \sin B \sin a;\\
\sin A \cos b &= \cos B \sin C + \sin B \cos C \cos a;\\
\sin A \sin c &= \sin C \sin a;\\
\sin A \cos c &= \sin B \cos C + \cos B \sin C \cos a;\\
\cos A &= -\cos B \cos C + \sin B \sin C \cos a.
\end{aligned}\right\}\qquad (9)$$

Applying these formulæ to each of the angles in succession, we have a set of equations by which, when two angles and the included side are given, the remaining parts may be found. The following are the remaining formulæ obtained by permutation:

$$\left.\begin{aligned}
\sin B \sin c &= \sin C \sin b;\\
\sin B \cos c &= \cos C \sin A + \sin C \cos A \cos b;\\
\sin B \sin a &= \sin A \sin b;\\
\sin B \cos a &= \sin C \cos A + \cos C \sin A \cos b;\\
\cos B &= -\cos C \cos A + \sin C \sin A \cos b.
\end{aligned}\right\}\qquad (10)$$

$$\left.\begin{aligned}
\sin C \sin a &= \sin A \sin c;\\
\sin C \cos a &= \cos A \sin B + \sin A \cos B \cos c;\\
\sin C \sin b &= \sin B \sin c;\\
\sin C \cos b &= \sin A \cos B + \cos A \sin B \cos c;\\
\cos C &= -\cos A \cos B + \sin A \sin B \cos c.
\end{aligned}\right\}\qquad (11)$$

105. If in the three sets of equations (6) to (8) we divide the second equation by the first, and the fourth by the third, and clear of denominators, we shall have the following additional equations:

$$\left.\begin{aligned}
\cot B \sin A &= \cot b \sin c - \cos c \cos A;\\
\cot C \sin A &= \cot c \sin b - \cos b \cos A;\\
\cot C \sin B &= \cot c \sin a - \cos a \cos B;\\
\cot A \sin B &= \cot a \sin c - \cos c \cos B;\\
\cot A \sin C &= \cot a \sin b - \cos b \cos C;\\
\cot B \sin C &= \cot b \sin a - \cos a \cos C.
\end{aligned}\right\}\qquad (12)$$

Treating the equations (9) to (11) in the same way, we shall get a similar set of equations; but on examination they will be found to differ from the equations (12) only in the arrangement of their terms, so that they need not be written.

106. Although the foregoing equations need to be transformed for most of their uses, they may in many cases be applied directly to the solution of spherical triangles. This is especially the case when two sides and the included angle, or two angles and the included side, are given and one or more of the remaining parts required. If, for instance, A, B, and c are given, we may from are the first two of equations (11) obtain the values of C and a by Prob. I. Chap. V. Then from the next two we may obtain c and C, and from the last $\cos C$, and thus a third value of C. If the work is correct, these three values of C will all agree.

Example. Two of the face-angles of a trihedral angle are $a = 132° 46'.7$ and $b = 59° 50'.1$, and the included edge-angle is $C = 56° 28'.4$. Find the remaining parts.

The computation of A, B, and c may be effected by the equations (8), as follows:

(1) $\cos b$,	9.701 13
(2) $\sin a$,	9.865 69
(3) $\sin C$,	9.920 97
(4) $\sin b$,	9.936 81
(5) $\cos a$,	– 9.831 98
(12) $\cos a \sin b$,	– 9.768 79
(13) $-\sin a \cos b \cos C$,	– 9.309 01
(14) Diff.,	0.459 78
(15) Add. log.,	0.589 12
(2) + (3) $\sin c \sin A$,	9.786 66
(13)+(15) $\sin c \cos A$,	– 9.898 13
(18) $\cos A$,	9.898 14
(16) $\tan A$,	– 9.888 53
(17) A,	142° 16′.4
(19) $\sin c$,	9.999 99
(31) Subtr. log.,	8.431 7
(28) $\cos a \cos b$,	– 9.533 11
(29) $\sin a \sin b \cos C$,	9.544 69

(6) $\sin b$,	9.936 81
(7) $\cos C$,	9.742 19
(8) $\cos b$,	9.701 13
(9) $\sin a$,	9.865 69
(11) $\sin b \cos C$,	9.679 00
(10) $\cos a$,	– 9.831 98
(20) $\sin a \cos b$,	9.566 82
(21) $-\cos a \sin b \cos C$,	+ 9.510 98
(22) Diff.,	0.055 84
(23) Add. log.,	0.329 84
(26) $\sin B$,	9.857 80
(3)+(4) $\sin c \sin B$,	9.857 78
(21)+(23) $\sin c \cos B$,	9.840 82
(24) $\tan B$,	0.016 96
(25) B,	46° 7′.1
(27) $\sin c$,	9.999 98
(30) Diff.,	0.011 58
$\cos c$,	7.964 8
c,	89° 28′.3

NOTE.—We may omit the "log" from the designation of the logarithms of the trigonometric functions whenever no uncertainty will thus arise.

In this computation the numbers in parentheses show the order in which the lines may be written. Lines (15), (23), and (31) are the addition or subtraction logarithms, from the tables for finding the logarithm of the sum or difference of two numbers which are given by their logarithms. The student can equally well find the numbers, add them together, and take the logarithm of their sum.

The agreement of the two values of sin C with each other and with cos C shows the correctness of the calculation.

107. The following transformation, similar to that of Prob. IV. Chap. V., will often render the work convenient. In the second and last of equations (11) let us put

$$\left.\begin{aligned} k \sin K &= \sin A \cos c; \\ k \cos K &= -\cos A. \end{aligned}\right\} \qquad (a)$$

By substitution these equations will then become

$$\left.\begin{aligned} \sin C \cos a &= -k \cos K \sin B + k \sin K \cos B \\ &= k \sin (K - B); \\ \cos C &= k \cos K \cos B + k \sin K \sin B \\ &= k \cos (K - B). \end{aligned}\right\} \qquad (b)$$

To apply these equations we compute k and K from (a), and then sin C cos a and cos C from (b). We complete the work by computing sin C sin a from the first equation of (11).

We may also transform the fourth equation by computing h and H from the equations

$$\begin{aligned} h \sin H &= \sin B \cos c; \\ h \cos H &= -\cos B. \end{aligned}$$

We shall then have

$$\begin{aligned} \sin C \cos b &= h \sin (H - A); \\ \cos C &= h \cos (H - A). \end{aligned}$$

To transform the equations (8) on the same plan, we may compute k, K, h and H from

$$\left.\begin{aligned} k \sin K &= \sin a \cos C; \\ k \cos K &= \cos a; \\ h \sin H &= \sin b \cos C; \\ h \cos H &= \cos b. \end{aligned}\right\} \qquad (c)$$

We then have, in the same way as before,

$$\sin c \cos A = k \sin (b - K); \qquad (d)$$
$$\sin c \cos B = h \sin (a - H);$$
$$\cos C = k \cos (b - K) = h \cos (a - H).$$

We compute the same example as before by these formulæ, as follows:

$\sin a$,	9.865 69
$\cos C$,	9.742 19
$\sin b$,	9.936 81
$k \sin K$,	9.607 88
$k \cos K$,	− 9.831 98
$\cos K$,	− 9.933 82
$\tan K$,	− 9.775 90
K,	149° 10′.0
b,	59° 50′.1
$h \sin H$,	9.679 00
$h \cos H$,	9.701 13
$\cos H$,	9.860 27
$\tan H$,	9.977 87
H,	43° 32′.4
a,	132° 46′.7
$\cos c$,	7.965 1
$\cos c$,	7.964 5
c,	89° 28′.3
B,	46° 7′.1

$\sin a$,	9.865 69
$\sin C$,	9.920 97
$\sin b$,	9.936 81
$b - K$ −	89° 19′.9
$a - H$,	89° 14′.3
$\sin (b - K)$,	− 9.999 97
$\log k$,	9.898 16
$\cos (b - K)$,	8.066 9
$\sin c \sin A$,	9.786 66
$\sin c \cos A$,	− 9.898 13
$\cos H$,	− 9.898 14
$\tan A$,	− 9.888 53
A,	142° 16′.4
$\sin c$,	9.999 99
$\sin (a - H)$,	9.999 96
$\log h$,	9.840 86
	8.123 6
$\sin c \sin B$,	9.857 78
$\sin c \cos B$,	9.840 82
$\tan B$,	0.016 96

EXERCISES.

1. Transform the equations (6), (7), (9), and (10) in the same way that we have transformed (8) and (11).

2. From the values of A, B, and c, which we have obtained in the last example, find those of a, b, and C with which we started.

3. If m be the arc joining the vertex A to the opposite side, prove

$$\cos b + \cos c = 2 \cos \tfrac{1}{2}a \cos m.$$

CHAPTER II.

RIGHT AND QUADRANTAL TRIANGLES.

Fundamental Definitions and Theorems.

108. *Def.* A right spherical triangle is one which has a right angle.

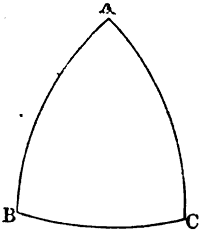
Def. A quadrantal spherical triangle is one which has a side equal to a quadrant.

Def. A trirectangular triangle is one which has three right angles.

Def. A birectangular triangle is one which has two right angles.

Def. A biquadrantal triangle is one which has two sides equal to a quadrant.

Theorem I. *Every birectangular triangle is also biquadrantal.*



Proof. Let ABC be a spherical triangle in which angle B = angle $C = 90°$. Then:

Because angle B is a right angle, the pole of the great circle BC is on the great circle BA.

Because angle C is a right angle, this pole is on the great circle CA. (*Geom.*)

Therefore the pole of BC is on both BA and CA, and therefore at their point of intersection A.

Because A is the pole of BC, AB and AC are quadrants.

Q.E.D.

Theorem II. Conversely, *Every biquadrantal triangle is also birectangular.*

Proof. Because every point of the polar circle of the point A is a quadrant distant from A, and because AB and AC are quadrants, this polar circle must pass through both B and C.

But only one great circle can pass through these points.

Therefore BC is the polar circle of A, and A the pole of BC.

Therefore the great circles AB and AC intersect BC at right angles. Q.E.D.

Cor. Every trirectangular triangle has three quadrants for its sides; and,

Conversely, *Every triangle having three quadrants for its sides is trirectangular.*

THEOREM III. *In a birectangular triangle the oblique angle is equal to its opposite side.*

Proof. Because the plane of the great circle BC intersects the planes of AB and of AC at right angles, the arc BC measures the dihedral angle between the planes AB and BC.

But the angle A is equal to this same dihedral angle.

Therefore $BC =$ angle A.

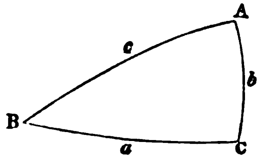
THEOREM IV. *The polar triangle of a right triangle is a quadrantal triangle.*

This follows at once from the fact that the angles of the one triangle are the supplements of the sides of the other.

EXERCISE.

Let the student translate the preceding definitions and theorems into those relating to the face- and edge-angles of a trihedral angle, and, which is the same thing, into those relating to the angles between three lines emanating from a point and the angles between their planes.

109. *Formulæ for right triangles.* Since in a right triangle one of the parts, the right angle, is known in advance, if two other parts be given the remaining three parts may be found.



An equation must therefore exist by which, when any two parts are given, any one of the three remaining parts may be found; hence between every combination of three parts out of the five there must be an equation. The number of combinations of 3 in 5 being $\frac{5 \cdot 4}{1 \cdot 2} = 10$, there must be ten such equations.

To find these equations let C be the right angle, and therefore c the hypothenuse. We seek for those equations in the sets (6) to (12) of the last chapter, in which the angle C enters, and in which the equation contains only three different parts. We then suppose

$$\sin C = 1;$$
$$\cos C = 0;$$
$$\cot C = 0.$$

The set from which the required equations are taken, the number in the set, and the result are shown as follows:

$$\left.\begin{array}{lll} \text{From} & (6)_3, & \sin a = \sin c \sin A; \quad (1) \\ \text{“} & (7)_1, & \sin b = \sin c \sin B; \quad (2) \end{array}\right\}$$
$$\left.\begin{array}{lll} \text{“} & (6)_4, & \cos A = \tan b \cot c; \quad (3) \\ \text{“} & (7)_2, & \cos B = \tan a \cot c; \quad (4) \end{array}\right\}$$
$$\begin{array}{lll} \text{“} & (8)_6, & \cos c = \cos a \cos b; \quad (5) \end{array}$$
$$\left.\begin{array}{lll} \text{“} & (9)_2, & \cos B = \cos b \sin A; \quad (6) \\ \text{“} & (9)_6, & \cos A = \cos a \sin B; \quad (7) \end{array}\right\}$$
$$\begin{array}{lll} \text{“} & (11)_5, & \cos c = \cot A \cot B; \quad (8) \end{array}$$
$$\left.\begin{array}{lll} \text{“} & (12)_5, & \cot A = \cot a \sin b; \quad (9) \\ \text{“} & (12)_6, & \cot B = \cot b \sin a. \quad (10) \end{array}\right\}$$

These ten equations will be found to include all combinations of three out of the five parts a, b, c, A, B. From each of them we may determine any one part in terms of the other two; for example, the first equation gives not only

$$\sin a = \sin c \sin A,$$

but

$$\sin c = \frac{\sin a}{\sin A},$$

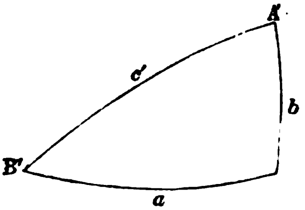
and

$$\sin A = \frac{\sin a}{\sin c}.$$

Properly speaking, only six of these equations are really distinct, as the other four can be derived from them by a mere interchange of letters between corresponding parts. For instance, since the same relation must hold between each oblique angle and its opposite side, the second equation may be derived from the first.

The equations which are thus related are connected by braces in the formulæ above.

110. *Napier's rules.* The six preceding formulæ, which may be found difficult to remember, have been included by Napier in two precepts of remarkable simplicity, and easily remembered.



Let us take for the five parts the sides a and b as before, and, instead of the other three parts, the *complements* of the oblique angles and of the hypothenuse. The fact that the complements are understood is indicated by accenting the letters in the diagram. We suppose

$$B' = 90° - B; \qquad c' = 90° - c; \qquad A' = 90° - a.$$

Omitting the right angle, the five parts a, b, A', c', B' form a continuous series, B' being followed in regular order by a. Now if we select any three of these parts, one of two cases must occur. Either—

(1) The three parts all adjoin each other, as B', a, b; a, b, A', etc., or

(2) Two of the parts adjoin each other and the third is separated from each of them by the remaining intervening parts.

The middle part of the three in the first case, or the separated part in the second, is called the *middle part.*

In the first case the extreme parts of the three are called *adjacent parts.*

In the second case the adjoining parts are called *opposite parts.*

111. Napier's rules are:

I. *The sine of the middle part equals the product of the tangents of the adjacent parts.*

II. *The sine of the middle part equals the product of the cosines of the opposite parts.*

The concurrence of the vowel a in *tangent* and *adjacent*, and of the vowel o in *cosine* and *opposite*, will help in remembering the relations.

Examples. 1. Let the parts be the hypothenuse and the two adjacent angles, or c, A and B.

The middle part is c', and A' and B' are adjacent parts.

By the rule,

$$\sin (90° - C) = \tan (90° - A) \tan (90° - B),$$

or

$$\cos c = \cot A \cot B,$$

agreeing with the formula (8).

2. Let the parts be a, A and C. The middle part is then a, and A' and C' are opposite parts. Therefore

$$\begin{aligned}\sin a &= \cos (90° - A) \cos (90° - C)\\ &= \sin A \sin c,\end{aligned}$$

agreeing with the formula (1).

3. Let the three parts be the two sides containing the right angle and one of the oblique angles, say a, b, and A. Then b is the middle part, and the other two adjacent parts. Therefore

$$\sin b = \tan a \tan (90° - A),$$

or

$$\sin b \cot a = \cot A,$$

agreeing with the ninth formula.

Exercises.

1. Given $A = 62° 29'.3$, $b = 25° 58'.8$; find a.
2. Given $B = 35° 29'.6$, $a = 75° 5'.3$; find A and b.
3. Given $a = 43° 40'.5$, $c = 98° 29'.1$; find A, B, and b.
4. Given $a = 148° 28'.2$, $A = 101° 3'.9$; find b and c.
5. Given $A = 50° 0'.8$, $B = 79° 57'.3$; find a, b, and c.

112. *Relations between four parts.* Although the preceding formulæ enable us, when two parts are given, to find the remaining three parts, each part has to be found independently by different equations. If all three parts are required, we may determine two of them by a single connected set of operations. For this purpose we select the appropriate equations from the sets (6) to (11) of the preceding chapter, choosing only those in which the angle C enters the second member.

113. Case I. Given *an angle and the adjacent side.*

When $C = 90°$, the last three equations (9) of § 104 are

$$\left.\begin{aligned}\sin A \sin c &= \sin a;\\ \sin A \cos c &= \cos a \cos B;\\ \cos A &= \cos a \sin B.\end{aligned}\right\} \qquad (11)$$

From the first two equations we obtain $\sin A$ and c. Since we thus have separate values of $\sin A$ and $\cos A$, the agreement of the two values of A serves as a check upon the accuracy of the computation.

If b is also required, the first two equations (9) of § 104 give

$$\left.\begin{aligned}\sin A \sin b &= \sin a \sin B;\\ \sin A \cos b &= \cos B.\end{aligned}\right\} \tag{12}$$

From which we obtain b and another value of $\sin A$.

By simply reversing equations (11) we obtain a and B when A and c are given.

Example. Given $a = 75°\ 5'.3$, $B = 35°\ 29'.6$, to find the remaining three parts.

$\sin a$,	9.985 12	
$\sin B$,	9.763 88	
$\cos a$,	9.410 49	
$\cos B$,	9.910 73	
$\sin A \sin c = \sin a$,	9.985 12	
$\sin A \cos c = \cos a \cos B$,	9.321 22	
$\tan c$,	0.663 90	$c = 77°\ 46'.0$
$\sin A$,	9.995 10	
$\cos A = \cos a \sin B$,	9.174 37	$A = 81°\ 24'.4$
$\sin A \sin b$,	9.749 00	
$\sin A \cos b$,	9.910 73	
$\tan b$,	9.838 27	$b = 34°\ 34'.2$
$\sin A$,	9.995 10	

EXERCISES.

1. Given $a = 34°\ 34'.2$, $B = 81°\ 24'.4$; find remaining parts.
2. Given $A = 45°\ 45'.4$, $c = 61°\ 49'.3$; find a and B.
3. Given $a = 120°\ 29'.6$, $B = 22°\ 59'.8$; find c and A.
4. Given $A = 98°\ 0'.4$, $b = 52°\ 7'.8$; find a and B.
5. Given $B = 133°\ 33'.7$, $a = 7°\ 29'.3$; find A and b.

114. CASE II. Given the two sides, a and b.

Putting $C = 90°$, the equations (8) of § 104 become

$$\left.\begin{aligned}\sin c\sin A &= \sin a;\\ \sin c\cos A &= \cos a\sin b;\\ \cos c &= \cos a\cos b;\\ \sin c\sin B &= \sin b;\\ \sin c\cos B &= \sin a\cos b.\end{aligned}\right\}\qquad(13)$$

The values of c and A are determined from the first three equations; those of c and B from the last three. The agreement of the two values of sin c with the one value of cosine c affords a check upon the accuracy of the work.

Exercises.

1. Given $a = 39° 6'.8$, $b = 82° 39'.6$; find A, B, c.
2. Given $a = 103° 40'.2$, $b = 62° 29'.3$; find A, B, c.
3. Given $a = 172° 1'.5$, $b = 158° 58'.8$; find A, B, c.

115. Case III. Given the hypothenuse and one angle.

The first three equations of Case I. and the first three of Case II. give

$$\left.\begin{aligned}\cos a\sin B &= \cos A;\\ \cos a\cos B &= \sin A\cos c;\\ \sin a &= \sin A\sin c;\\ \cos a\sin b &= \cos A\sin c;\\ \cos a\cos b &= \cos c.\end{aligned}\right\}\qquad(14)$$

From which a, b, and B may be determined.

Exercises.

In a triangle, right-angled at C, prove the relations:

1. $\sin A\sin 2b = \sin c\sin 2B$.
2. $\sin 2A\sin c = \sin 2a\sin B$.
3. $\sin 2a\sin 2b = 4\cos A\cos B\sin^2 c$.
4. $\sin^2\frac{1}{2}c = \sin^2\frac{1}{2}a\cos^2\frac{1}{2}b + \sin^2\frac{1}{2}b\cos^2\frac{1}{2}a$.
5. $\sin(c - b) = \tan^2\frac{1}{2}A\sin(b + c)$.
6. $\sin a\cos b = \tan\frac{1}{2}A\sin(b + c)$.
7. In a right triangle of which the oblique angles are

$$A = 69° 23'.7,\qquad B = 60° 7'.6,$$

find the length of the perpendicular from the right angle upon the base, and the angles which it forms with the sides.

8. In a right triangle is given

$$c = 75^\circ 25', \qquad a = 52^\circ 16';$$

find the lengths of the segments into which a is divided by the bisector of A.

EXERCISES IN GEOMETRIC APPLICATION.

9. From a point P above a plane an oblique line PO is drawn, meeting the plane in O and making the angle A with the plane. Let Q be the projection of P upon the plane, so that OQ is the projection of OP. Through O a line OM is drawn, making an angle $QOM = B$ with the projection OQ. It is required to express the angle POM in terms of A and B.

Ans. $\text{Cos } POM = \cos A \cos B$.

10. In the preceding case, if a perpendicular PS be dropped from P upon OM, express the length OS in terms of the angle A and B and the length OP. *Ans.* $OS = OP \cos A \cos B$.

11. Two planes intersect at right angles along a line I. From any point R of I one line is drawn in each plane, making the respective angles A and B with I. Express the angle C between these lines. *Ans.* $\text{Cos } C = \cos A \cos B$.

12. Two planes intersecting at right angles along a line I are intersected by a third plane, making with them the respective angles P and Q. Express the angles which the three lines of intersection make with each other.

Ans. If we put PI for the angle between I and that edge along which the dihedral angle P is formed, etc., we have

$$\cos PI = \frac{\cos Q}{\sin P};$$

$$\cos QI = \frac{\cos P}{\sin Q};$$

$$\cos PQ = \cot P \cot Q.$$

116. *Isosceles triangles.* An isosceles spherical triangle may be divided into two symmetrical right triangles by a perpendicular from its vertex upon its base. If we put

c, each of the equal sides;

C, each of the equal angles at the base;

b, the base, or third side;
B, the angle at the vertex;
P, the middle point of b;
p, the length of the perpendicular BP from B upon b,—

we shall then have two right triangles in each of which the oblique angles are C and $\frac{1}{2}B$, the hypothenuse is c, and the sides containing the right angle are p and $\frac{1}{2}b$. The equations of § 109 will then give

$$\begin{aligned} \sin \tfrac{1}{2}b &= \sin c \sin \tfrac{1}{2}B; \\ \sin p &= \sin c \sin C; \\ \text{etc.} & \qquad \text{etc.} \end{aligned}$$

EXERCISES.

1. The equal sides of an isosceles triangle are each 45°, and the angle which they contain is 95°. Find the base and the angles at the base.

2. If the base of an isosceles triangle is 95°, and the angles at the base each 45°, find the remaining parts.

117. *Quadrantal triangles.* Since the polar of a right triangle is a quadrantal triangle, the formulæ for quadrantal triangles may be obtained by applying the formulæ of § 109 to the polar triangle. The side c will then be a quadrant, and the relations among the other parts will be

$$\left.\begin{aligned} \sin A &= \sin C \sin a; \\ \cos a &= -\tan B \cot C; \\ \cos C &= -\cos A \cos B; \\ \cos b &= \cos B \sin a; \\ \cos C &= -\cot a \cot b; \\ \sin B &= \tan A \cot a. \end{aligned}\right\} \qquad (15)$$

If we take, as the five parts of the triangle,

$$A,\ B,\ 90° - a,\ 90° - b,\ C - 90°, \qquad (a)$$

and omit the hypothenuse c, the above formulæ will be expressed by a set of rules identical in expression with those of Napier. For example, let us consider the parts a, b, C. Here C will be a middle part, and a and b adjacent parts. Applying Napier's rules to this case, with the parts (a) we have

$$\sin (C - 90^\circ) = \tan (90^\circ - a) \tan (90^\circ - b);$$

which gives $$-\cos C = \cot a \cot b,$$

an equation identical with the fifth of the above list.

EXERCISES.

1. Let the student deduce the six equations (15) by applying Napier's rules to the parts (a).

2. Through the same point there pass two lines intersecting at right angles, and a plane P making the angle α with one of the lines, and the angle β with the other. Express the angle which the plane P forms with the plane of the lines.

Ans. $\sin A = \sqrt{\sin^2 \alpha + \sin^2 \beta}$.

3. The sides of an obelisk have a slope of 8° from the perpendicular. What is the face-angle at the base of the obelisk, the slope of the edges, and the dihedral angle between two adjacent lateral faces? *Ans.* Face-angle at base, 82° 4′.6; slope, 11° 14′.5; dihedral angle, 91° 6′.6.

In this problem, to reduce to a spherical triangle, consider the centre of the sphere to be at a corner of the obelisk. The slope of the edge will not be represented by either of the six parts of the triangle, but by the complement of the perpendicular from the vertex upon the base.

4. A mason cuts a stone with a rectangular base and four lateral edges, each making an angle of 60° with the base at its corners. What is the inclination of each lateral face to the base, and the dihedral angle between the faces, supposing such inclinations and dihedral angles all equal? *Ans.* 67° 47′.5 and 98° 12′.8.

5. In another stone the base is rectangular; one lateral face makes an angle of 68° 29′ with the base, and the lateral edge bounding this face makes an angle of 52° 15′ with the base. What angles does the adjacent lateral face make with the first face and with the base?

6. When the angular distance of the sun from the south point of the horizon is 75°, and from the west point 60°, what is its altitude above the horizon?

CHAPTER III.

TRANSFORMATION OF THE FORMULÆ OF SPHERICAL TRIGONOMETRY.

118. Although the formulæ already given suffice for the solution of every spherical triangle, there are many transformations which will facilitate the applications of spherical trigonometry, and render the solutions of triangles more accurate and convenient.

Let us first take the fundamental equation (1) of Chapter I.,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

We may express this in the form

$$\left.\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c - \sin b \sin c\,(1 - \cos A)\\ &= \cos (b - c) - 2 \sin b \sin c \sin^2 \tfrac{1}{2}A.\end{aligned}\right\}(1)$$

Moreover, because

$$-2 \sin b \sin c = \cos (b + c) - \cos (b - c), \qquad (\S\ 43)$$

we have, by substituting,

$$\left.\begin{aligned}\cos a &= \cos (b - c)\,(1 - \sin^2 \tfrac{1}{2}A) + \cos (b + c) \sin^2 \tfrac{1}{2}A\\ &= \cos (b - c) \cos^2 \tfrac{1}{2}A + \cos (b + c) \sin^2 \tfrac{1}{2}A.\end{aligned}\right\}(2)$$

This last equation may also be derived by the following elegant process. The original fundamental equation may be written

$$\begin{aligned}\cos a = \cos b \cos c\,&(\cos^2 \tfrac{1}{2}A + \sin^2 \tfrac{1}{2}A)\\ + \sin b \sin c\,&(\cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A)\end{aligned}$$

(because $\cos^2 \frac{1}{2}A + \sin^2 \frac{1}{2}A = 1$, and $\cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A = \cos A$).

By conjoining the coefficients of $\cos^2 \frac{1}{2}A$ and of $\sin^2 \frac{1}{2}A$, the equation (2) follows by the addition theorem.

By a similar process, from the equation

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

we obtain

$$\cos A = -\cos (B + C) - 2 \sin B \sin C \sin^2 \tfrac{1}{2}a; \qquad (3)$$

$$\cos A = -\cos (B + C) \cos^2 \tfrac{1}{2}a - \cos (B - C) \sin^2 \tfrac{1}{2}a. \qquad (4)$$

By a slight modification of the process employed in forming the equations (1) and (3) we may find

$$\cos a = \cos(b + c) + 2 \sin b \sin c \cos^2 \tfrac{1}{2}A; \qquad (5)$$
$$\cos A = -\cos(B - C) + 2 \sin B \sin C \cos^2 \tfrac{1}{2}a; \qquad (6)$$

which equations the student may prove as an exercise.

119. *Expressions when three sides or three angles are given.* From the last equation (1) we find

$$\sin^2 \tfrac{1}{2}A = \frac{\cos(b - c) - \cos a}{2 \sin b \sin c};$$

by which any angle is expressed in terms of the three sides.

By § 44, 13 we have

$$\cos(b - c) - \cos a = 2 \sin \tfrac{1}{2}(a + c - b) \sin \tfrac{1}{2}(a + b - c). \qquad (a)$$

If we put s for half the sum of the sides, namely,

$$s = \tfrac{1}{2}(a + b + c),$$

we have

$$\left.\begin{aligned} \tfrac{1}{2}(a + c - b) &= s - b; \\ \tfrac{1}{2}(a + b - c) &= s - c. \end{aligned}\right\} \qquad (b)$$

Substituting these values in (a), the expression for $\sin^2 \tfrac{1}{2}A$ becomes

$$\sin^2 \tfrac{1}{2}A = \frac{\sin(s - b) \sin(s - c)}{\sin b \sin c};$$

then, by permutation,

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2}B &= \frac{\sin(s - c) \sin(s - a)}{\sin c \sin a}; \\ \sin^2 \tfrac{1}{2}C &= \frac{\sin(s - a) \sin(s - b)}{\sin a \sin b}. \end{aligned}\right\} \qquad (7)$$

To find similar expressions for the cosines we take equation (5), which gives

$$\cos^2 \tfrac{1}{2}A = \frac{\cos a - \cos(b + c)}{2 \sin b \sin c}.$$

But

$$\cos a - \cos(b + c) = 2 \sin \tfrac{1}{2}(b + c + a) \sin \tfrac{1}{2}(b + c - a)$$
$$= 2 \sin s \sin(s - a) \qquad \text{[from } (b)\text{]}.$$

Therefore

$$\left.\begin{aligned} \cos^2 \tfrac{1}{2}A &= \frac{\sin s \sin(s - a)}{\sin b \sin c}; \\ \cos^2 \tfrac{1}{2}B &= \frac{\sin s \sin(s - b)}{\sin c \sin a}; \\ \cos^2 \tfrac{1}{2}C &= \frac{\sin s \sin(s - c)}{\sin a \sin b}. \end{aligned}\right\} \qquad (8)$$

Since an angle near 90° cannot be accurately determined by its sine, nor one near 0° by its cosine, neither of the formulæ (7) or (8) can be advantageously used in all cases. But by taking the quotient of each equation (7) by the corresponding one of (8) we have

$$\left.\begin{aligned}\tan^2 \tfrac{1}{2}A &= \frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)};\\ \tan^2 \tfrac{1}{2}B &= \frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)};\\ \tan^2 \tfrac{1}{2}C &= \frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}.\end{aligned}\right\} \qquad (9)$$

120. Treating the equations (3) and (6) in the same way, and putting

$$S = \tfrac{1}{2}(A + B + C),$$

we find the following expressions for the sides, in terms of the three angles:

$$\left.\begin{aligned}\sin^2 \tfrac{1}{2}a &= -\frac{\cos S\cos(S-A)}{\sin B\sin C};\\ \sin^2 \tfrac{1}{2}b &= -\frac{\cos S\cos(S-B)}{\sin C\sin A};\\ \sin^2 \tfrac{1}{2}c &= -\frac{\cos S\cos(S-C)}{\sin A\sin B}.\end{aligned}\right\} \qquad (10)$$

$$\left.\begin{aligned}\cos^2 \tfrac{1}{2}a &= \frac{\cos(S-B)\cos(S-C)}{\sin B\sin C};\\ \cos^2 \tfrac{1}{2}b &= \frac{\cos(S-C)\cos(S-A)}{\sin C\sin A};\\ \cos^2 \tfrac{1}{2}c &= \frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}.\end{aligned}\right\} \qquad (11)$$

$$\left.\begin{aligned}\tan^2 \tfrac{1}{2}a &= -\frac{\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)};\\ \tan^2 \tfrac{1}{2}b &= -\frac{\cos S\cos(S-B)}{\cos(S-C)\cos(S-A)};\\ \tan^2 \tfrac{1}{2}c &= -\frac{\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}.\end{aligned}\right\} \qquad (12)$$

For the solution of a triangle in which all three sides or all three angles are given, the equations (9) and (12) are preferable.

For convenience in computation the following slight modification may be made. Put

$$p = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$$

then from (9),

$$\tan^2 \tfrac{1}{2}A = \frac{p^2}{\sin^2(s-a)};$$

extracting the square root, and writing the remaining equations,

$$\left.\begin{aligned} \tan \tfrac{1}{2}A &= \frac{p}{\sin(s-a)}; \\ \tan \tfrac{1}{2}B &= \frac{p}{\sin(s-b)}; \\ \tan \tfrac{1}{2}C &= \frac{p}{\sin(s-c)}. \end{aligned}\right\} \qquad (13)$$

In the same way, if we put

$$P = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}};$$

we find, from (12),

$$\left.\begin{aligned} \tan \tfrac{1}{2}a &= P\cos(S-A); \\ \tan \tfrac{1}{2}b &= P\cos(S-B); \\ \tan \tfrac{1}{2}c &= P\cos(S-C). \end{aligned}\right\} \qquad (14)$$

EXAMPLE OF COMPUTATION.

Given the three sides,

$a = 76° 29'.3$, $b = 93° 18'.6$, $c = 122° 7'.7$;

find the angles.

$a =$ 76° 29′.3	$s =$ 145° 57′.8	cosec 0.252 02
$b =$ 93° 18′.6	$s - a =$ 69° 28′.5	sin 9.971 52
$c =$ 122° 7′.7	$s - b =$ 52° 39′.2	9.900 36
$2s =$ 291° 55′.6	$s - c =$ 23° 50′.1	9.606 49
		$\log p^2$, 9.730 39
9.865 20	9.865 20	$\log p$, 9.865 20
$\sin(s-a)$, 9.971 52	$\sin(s-b)$, 9.900 36	$\sin(s-c)$, 9.606 49
$\tan \frac{1}{2}A$, 9.893 68	$\tan \frac{1}{2}B$, 9.964 84	$\tan \frac{1}{2}C$, 0.258 71
$\frac{1}{2}A$, 38° 3′.35	$\frac{1}{2}B$, 42° 41′.0	$\frac{1}{2}C$, 61° 8′.3
A, 76° 6′. 7	B, 85° 22′.0	C, 122° 16′.6

We may now check the results by computing the values of $\frac{\sin a}{\sin A}$, $\frac{\sin b}{\sin B}$, $\frac{\sin c}{\sin C}$, and seeing whether they agree. We find

$$\begin{array}{lll}
\sin a = 9.987\,81; & \sin b = 9.999\,27; & \sin c = 9.927\,81 \\
\sin A = \underline{9.987\,11}; & \sin B = \underline{9.998\,58}; & \sin C = \underline{9.927\,10} \\
\qquad\quad .000\,70 & \qquad\quad .000\,69 & \qquad\quad .000\,71
\end{array}$$

Although the agreement is not perfect, the deviations are no greater than those due to the omission of decimals in the logarithms and angles. The angle B is that for which the check is most doubtful, because it is so near 90° that it may be changed 1′ or 2′ without changing the last figures in log. sin. by more than one or two units in the fifth place. In such a case certainty can be reached only by a duplicate computation.

EXERCISES.

1. Given $a = 105°\ 6'.8$, $b = 93°\ 39'.9$, $c = 50°\ 20'.3$; find the angles.

2. Given $A = 46°\ 59'.3$, $B = 122°\ 32'.6$, $C = 139°\ 0'.3$; find the sides.

3. Given $A = 78°\ 40'.7$, $B = 102°\ 29'.5$, $C = 86°\ 49'.4$; find the sides.

4. If the sides of a spherical triangle are each 120°, find the following numerical expressions for the sine, cosine, and tangent of each of the angles, and thence, by the aid of the tables, find the angles themselves.

$$\text{sines} = \sqrt{\tfrac{2}{3}};$$
$$\text{cosines} = \sqrt{\tfrac{1}{3}};$$
$$\text{tangents} = \sqrt{2}.$$

121. *Cagnoli's equation.* Cagnoli's equation is

$$\sin a \sin b + \cos a \cos b \cos C = \sin A \sin B - \cos A \cos B \cos c. \quad (15)$$

Proof. Multiplying the third equation (§ 97, 1) by cos C, we have

$$\left.\begin{aligned}
\cos c \cos C &= \cos a \cos b \cos C + \sin a \sin b \cos^2 C \\
&= \cos a \cos b \cos C + \sin a \sin b - \sin a \sin b \sin^2 C.
\end{aligned}\right\}(a)$$

The equation of sines (§ 99, 2) gives

$$\sin a \sin C = \sin A \sin c;$$
$$\sin b \sin C = \sin B \sin c;$$

whence, by multiplying,

$$\sin a \sin b \sin^2 C = \sin A \sin B \sin^2 c$$
$$= \sin A \sin B - \sin A \sin B \cos^2 c.$$

Substituting this value in (a), and interchanging the terms of the equation,

$$\sin a \sin b + \cos a \cos b \cos C$$
$$= \cos c \cos C + \sin A \sin B - \sin A \sin B \cos^2 c$$
$$= \sin A \sin B + \cos c\,(\cos C - \sin A \sin B \cos c).$$

From the last equation (11), § 104, we have

$$\cos C - \sin A \sin B \cos c = - \cos A \cos B.$$

Making this substitution, we have the equation (15) as enunciated.

Of course two other similar equations may be obtained by permuting the symbols.

122. *Gauss's equations.* We write the four equations,

(a) Cagnoli's equation,

(b) and (c) the fundamental equations $(1)_3$ and $(3)_3$, and

(d) the identical equation $1 = 1$, as follows:

(a) $\sin a \sin b + \cos a \cos b \cos C = \sin A \sin B - \cos A \cos B \cos c$;

(b) $\cos a \cos b + \sin a \sin b \cos C = \cos c$;

(c) $\cos C = - \cos A \cos B + \sin A \sin B \cos c$;

(d) $1 = 1$.

Taking the sum of the four equations, and substituting

$$\cos (a - b) = \cos a \cos b + \sin a \sin b,$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B,$$

we have

$$\cos (a - b) + \cos C \cos (a - b) + \cos C + 1 =$$
$$- \cos (A + B) - \cos c \cos (A + B) + \cos c + 1;$$

or

$$(1 + \cos C)\,[1 + \cos (a - b)] = (1 + \cos c)\,[1 - \cos (A + B)] \quad (e)$$

If we form the sum $-(a) - (b) + (c) + (d)$, and reduce in the same way, we find

$$(1 + \cos C)\,[1 - \cos (a - b)] = (1 - \cos c)\,[1 - \cos (A - B)] \quad (f)$$

The sum $-(a) + (b) - (c) + (d)$ gives

$$(1 - \cos C)\,[1 + \cos (a + b)] = (1 + \cos c)\,[1 + \cos (A + B)] \quad (g)$$

The sum $(a) - (b) - (c) + (d)$ gives

$$(1 - \cos C)\,[1 - \cos (a + b)] = (1 - \cos c)\,[1 + \cos (A - B)] \quad (h)$$

In the equations (e), (f), (g), (h) we substitute the values of $1 \pm \cos$, namely,

$$\begin{aligned} 1 + \cos C &= 2 \cos^2 \tfrac{1}{2}C; \\ 1 - \cos C &= 2 \sin^2 \tfrac{1}{2}C; \end{aligned}$$

etc. etc.,

and dividing by 2, we obtain

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2}C \sin^2 \tfrac{1}{2}(a + b) &= \sin^2 \tfrac{1}{2}c \cos^2 \tfrac{1}{2}(A - B); \ (h) \\ \sin^2 \tfrac{1}{2}C \cos^2 \tfrac{1}{2}(a + b) &= \cos^2 \tfrac{1}{2}c \cos^2 \tfrac{1}{2}(A + B); \ (g) \\ \cos^2 \tfrac{1}{2}C \sin^2 \tfrac{1}{2}(a - b) &= \sin^2 \tfrac{1}{2}c \sin^2 \tfrac{1}{2}(A - B); \ (f) \\ \cos^2 \tfrac{1}{2}C \cos^2 \tfrac{1}{2}(a - b) &= \cos^2 \tfrac{1}{2}c \sin^2 \tfrac{1}{2}(A + B). \ (e) \end{aligned}\right\} \quad (16)$$

Extracting the square roots, we have

$$\begin{aligned} \sin \tfrac{1}{2}C \sin \tfrac{1}{2}(a + b) &= \sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A - B); \\ \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(a + b) &= \cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A + B); \\ \cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a - b) &= \sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A - B); \\ \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(a - b) &= \cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A + B). \end{aligned}$$

In strictness, the second members of this equation may have the negative as well as the positive sign. But it is easy to show that when the sides and angles are all less than 180°, all the members of the equations are positive. Hence the positive sign is the only one necessary to be taken into account.

These equations are applicable when any three consecutive parts of the triangle—two angles and the included side, or two sides and the included angle—are given.

In the first case the three given parts are all in the right-hand members of the equations; in the second case they are all on the left.

These equations are written in the most convenient order for use in the first case; in the second, the following is the order and arrangement:

$$\left.\begin{aligned} \sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A - B) &= \cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a - b); \\ \sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A - B) &= \sin \tfrac{1}{2}C \sin \tfrac{1}{2}(a + b); \\ \cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A + B) &= \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(a - b); \\ \cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A + B) &= \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(a + b). \end{aligned}\right\} \quad (17)$$

These are commonly known as *Gauss's equations*, after Gauss, who first introduced them into astronomical computations. They had, however, been previously published anonymously by Delambre.

Example. Given $a = 132° 46'.7$, $b = 59° 50'.1$, $C = 56° 28'.4$, to find the remaining parts.

a,	132° 46′.7	$\sin \frac{1}{2}c \sin \frac{1}{2}(A - B)$,	+ 9.719 08
b,	59° 50′.1	$\sin \frac{1}{2}c \cos \frac{1}{2}(A - B)$,	+ 9.672 34
$a + b$,	192° 36′.8	$\tan \frac{1}{2}(A - B)$,	0.046 74
$a - b$,	72° 56′.6	$\sin \frac{1}{2}c$,	9.847 48
$\frac{1}{2}(a + b)$,	96° 18′.4	$\cos \frac{1}{2}c \sin \frac{1}{2}(A + B)$,	9.850 32
$\frac{1}{2}(a - b)$,	36° 28′.3	$\cos \frac{1}{2}c \cos \frac{1}{2}(A + B)$,	− 8.715 77
$\frac{1}{2}C$,	28° 14′.2	$\tan \frac{1}{2}(A + B)$,	− 1.134 55
		$\cos \frac{1}{2}c$,	9.851 49

$\sin \frac{1}{2}(a - b)$,	9.774 10	$\frac{1}{2}(A - B)$,	48° 4′.6
$\cos \frac{1}{2}C$,	9.944 98	$\frac{1}{2}(A + B)$,	94° 11′.7
$\cos \frac{1}{2}(a - b)$,	9.905 34	$\frac{1}{2}c$,	44° 44′.1
$\sin \frac{1}{2}(a + b)$,	+ 9.997 37	A,	142° 16′.3
$\sin \frac{1}{2}C$,	+ 9.674 97	B,	46° 7′.1
$\cos \frac{1}{2}(a + b)$,	− 9.040 80	c,	89° 28′.2

EXERCISES.

Compute the remaining three parts of each of the following triangles by Gauss's formulæ:

1. $A = 79° 28'.6$, $b = 28° 20'.3$, $c = 112° 1'.9$.
2. $A = 32° 58'.5$, $B = 65° 26'.7$, $c = 56° 21'.2$.
3. $a = 112° 12'.6$, $b = 124° 48'.2$, $C = 18° 17'.0$.
4. $a = 52° 22'.2$, $B = 160° 0'.8$, $C = 129° 52'.4$.

123. *Napier's analogies.* If, in the preceding problem, only the two remaining sides in the one case, or the two remaining angles in the other, are wanted, the process may be shortened.

Dividing the first of (16) by the second, and the third by the fourth, we have

$$\left.\begin{aligned} \tan \tfrac{1}{2}(a + b) &= \tan \tfrac{1}{2}c \, \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)}; \\ \tan \tfrac{1}{2}(a - b) &= \tan \tfrac{1}{2}c \, \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)}; \end{aligned}\right\} \qquad (18)$$

from which may be found a and b when A, B, and C are given.

In the same way we find, from (17),

$$\left.\begin{aligned}\tan\tfrac{1}{2}(A-B) &= \cot\tfrac{1}{2}C\,\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)};\\ \tan\tfrac{1}{2}(A+B) &= \cot\tfrac{1}{2}C\,\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)};\end{aligned}\right\} \qquad (19)$$

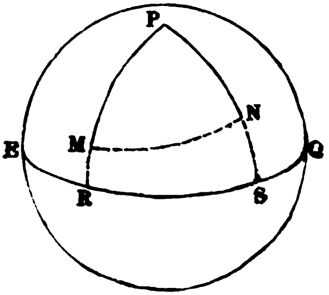
from which A and B may be found when a, b, and C are given.

The equations (18) and (19) are known as *Napier's analogies.*

CHAPTER IV.

MISCELLANEOUS APPLICATIONS.

124. *To find the distance between two points on the earth's surface and their direction from each other, when their latitudes and longitudes are given.**



Let M and N be the two points whose latitudes and longitudes are given; P, the pole; and EQ, the equator. Join M to N by the arc of a great circle. Also let

φ = the lat. and λ = the long. of M;
φ' = " " λ' = " " N.

If from the pole P we draw through M and N arcs of great circles PMR and PNS, meeting the equator in R and S, we shall have

$$\varphi = RM = \text{latitude of } M;$$
$$\varphi' = SN = \text{latitude of } N.$$

If we suppose PQ to be the meridian from which we reckon longitudes,

$$\text{Angle } QPM = \lambda;$$
$$\text{Angle } QPN = \lambda'.$$

Then because $PR = PS = 90°$, we have in the triangle MPN

$$PM = 90° - \varphi;$$
$$PN = 90° - \varphi';$$
$$\text{Angle } P = \lambda - \lambda'.$$

---

* It is assumed in the solution of this problem that the earth is a sphere. Although the assumption is not strictly correct, the error to which it can give rise can never amount to more than a few thousandths of the whole distance.

This is a case, therefore, in which we have given two sides and the included angle to find the remaining side. If we put d for the required distance, the general formula gives

$$\left.\begin{aligned}\cos d &= \cos PM \cos PN + \sin PM \sin PN \cos P\\ &= \sin \varphi \sin \varphi' + \cos \varphi \cos \varphi' \cos (\lambda - \lambda').\end{aligned}\right\} \qquad (a)$$

This equation will suffice to determine the distance between the points in arc of a great circle.

To reduce it to statute miles, the degrees must be multiplied by 69⅙; and to reduce to nautical miles, by 60.

To solve the problem completely we should know not only the distance but the direction, or the angle which the great circle joining the two points makes with the meridian. This angle is different at the two points, being equal to the angle M of the spherical triangle at one point, and to N at the other. Hence the complete solution requires the complete solution of the spherical triangle PMN, for which we may use the Gaussian equations instead of the form (*a*).

EXERCISES.

1. Find (1) the distance,* both in degrees and nautical miles, between New York and Liverpool, on an arc of a great circle; (2) the direction in which a ship would sail on leaving the one port for the other, on an arc of a great circle, and the direction in which she would be sailing on approaching her destination. The positions of the cities are:

New York........lat. + 40° 42′.7, long. 74° 0′.0 west.
Liverpool.........lat. + 53° 24′.1, long. 2° 59′.1 west.

2. Compute the distance between Liverpool and Rio de Janeiro, the position of the latter being:

Latitude, − 22° 54′.7.
Longitude, 43° 9′.0 west.

Note that the latitude of Rio is algebraically negative, being reckoned *south* from the equator.

3. If a ship sails from New York, starting due east, and continues her course on an arc of a great circle, what will be her latitude when she reaches the meridian of Greenwich, and in what direction will she then be sailing?

---

* By *distance*, as used here, distance on the arc of a great circle is to be understood, unless explicitly stated otherwise.

Geometrical Applications.

125. *Def.* Three straight lines, each perpendicular to the other two and all passing through a common point, are called a **system of rectangular axes.**

Def. The common point of intersection is called the **origin.**

Def. Three planes, each perpendicular to the other two, are called **three rectangular planes.**

Remark. It is shown in geometry that three rectangular planes intersect each other in lines forming a system of rectangular axes.

If a sphere have its centre in the common point of intersection of three rectangular planes, these planes will intersect its surface in three great circles, forming in all eight trirectangular triangles.

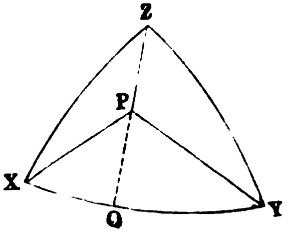
126. THEOREM. *The sum of the squares of the cosines of the three angles which a straight line forms with three rectangular axes is equal to unity.*

This theorem is expressed in trigonometric language thus:

If α, β, *and* γ *be the angles which a straight line forms with three rectangular axes*, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{1}$$

Proof. Let the line pass through the common point of intersection of the axes. Let a sphere have its centre at this point, and let X, Y, Z, and P be the points in which the axes and the line intersect the spherical surface.



Join PX, PY, and PZ by arcs of great circles, and produce ZP until it meets XY in Q. Then—

Because the angles α, β, and γ are formed at the centre of the sphere,

Arc PX measures angle α;
" PY " " β;
" PZ " " γ.

Because XYZ is a trirectangular triangle, Z is the pole of XY, and ZQ is therefore perpendicular to XY. Therefore the spherical triangles XQP and YQP are right-angled at Q, and, by § 109, 5,

$$\cos PX = \cos \alpha = \cos XQ \cos QP;$$
$$\cos PY = \cos \beta = \cos YQ \cos QP.$$

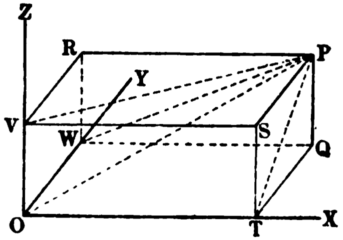
Taking the squares of these equations and adding them to the square of $\cos PZ = \cos \gamma$, we have

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \cos^2 QP(\cos^2 XQ + \cos^2 YQ) + \cos^2 PZ\\ &= \cos^2 QP + \cos^2 PZ \text{ (because } XY = 90°)\\ &= 1 \text{ (because } QZ = 90°). \quad \text{Q.E.D.}\end{aligned}$$

Another proof. Let OX, OY, and OZ be the axes meeting in O, and let OP be the line making with the lines OX, OY, OZ the respective angles

$$POX = \alpha;$$
$$POY = \beta;$$
$$POZ = \gamma.$$

Through P pass three planes parallel to the respective planes formed by the axes, taken two and two, namely,



Plane $PSVR$ || plane XOY;
Plane $PQTS$ || plane YOZ;
Plane $PRWQ$ || plane ZOX.

These planes will then form a rectangular parallelopiped $OTQW$-$VSPR$, of which OP will be a diagonal. By the property of this diagonal (Geom., § 692),

$$OP^2 = OT^2 + OW^2 + OV^2. \qquad (a)$$

Moreover, because these planes are respectively parallel to the three rectangular planes, and because each of the axes is perpendicular to one of these planes, the three planes in question are each perpendicular to one of the axes.

If we join PT, PW, and PV, these lines, being in planes which, as just shown, are perpendicular to the axes OX, OY, and OZ, will be perpendicular to these axes (Geom.), and we shall have

$$\left.\begin{aligned}OT &= OP \cos POX = OP \cos \alpha;\\ OW &= OP \cos POY = OP \cos \beta;\\ OV &= OP \cos POZ = OP \cos \gamma.\end{aligned}\right\} \qquad (b)$$

Taking the sum of the squares of these three equations, and substituting for $OT^2 + OW^2 + OV^2$ its value (a), we have

$$OP^2 = OP^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma).$$

Dividing by OP^2,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \quad \text{Q.E.D.} \qquad (2)$$

127. *Corollary.* It is easily shown that the angles α, β, and γ are the complements of the angles which OP forms with the three rectangular planes. For, pass a plane through P and OZ. Because $OZ \perp$ plane XOY, the cutting plane OZP is also perpendicular to XOY, and the line OQ in which it cuts $XOY \perp OZ$. But, by definition of the inclination of a line to a plane (Geom., § 603), the angle POQ will be the inclination of OP to the plane XOY. Therefore, if we call this inclination a, we have

$$a + \alpha = 90^\circ;$$

and in the same way,

$$b + \beta = 90^\circ;$$

$$c + \gamma = 90^\circ;$$

b and c being the inclinations of the line to the two other planes. Therefore $\sin a = \cos \alpha$, $\sin b = \cos \beta$, and $\sin c = \cos \gamma$; whence

$$\sin^2 a + \sin^2 b + \sin^2 c = 1. \qquad (3)$$

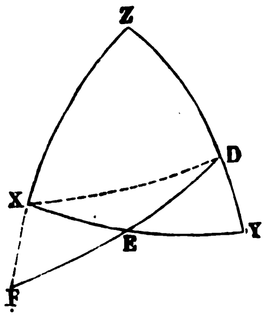
Because parallel lines have equal inclinations to a plane, the angles which any straight line makes with the three planes are equal to those made by a parallel to that line through O. Hence:

The sum of the squares of the sines of the three angles which any straight line makes with three rectangular planes is equal to unity.

128. *Case of a plane cutting three rectangular planes.* We have the following theorem:

THEOREM. *If any plane intersect three rectangular planes, the sum of the squares of the cosines of the three angles which it forms with them is equal to unity.*

This can be demonstrated from the preceding theorem by dropping a perpendicular from the common point of intersection of the three rectangular planes upon the cutting plane. Because the rectangular axes and this line are each perpendicular to one of the four planes, the angles which they form with each other are equal to the angles between the planes. The theorem, having already been proved for the angles formed by the lines, must therefore be true of the equal angles formed by the planes.



But the theorem may also be proved independently as follows:

Pass a sphere around the origin as a centre, and let XY, YZ, ZX be the arcs of great circles in which the rectangular planes intersect its surface.

Pass a plane through the centre parallel to the intersecting plane, and let FD be the great circle in which it intersects the spherical surface.

Put

$$\left.\begin{array}{l}\text{Angle } YDE = \alpha;\\ \text{Angle } XFE = \beta;\\ \text{Angle } XED = \gamma;\end{array}\right\}\ \begin{array}{l}\text{the three angles which the in-}\\ \text{tersecting plane forms with}\\ \text{the three rectangular planes.}\end{array}$$

Join XD by an arc of a great circle. Then—

Because $XY = XZ = 90°$, X is the pole of YZ, and

$$XD = 90°;$$

$$\text{Angle } ADC = \text{angle } ADB = 90°;$$

Because AFD and AED are triangles of which the side AD is 90°,

$$\cos AFD = \cos\beta = -\cos FAD \cos EDA;$$

$$\cos AED = \cos\gamma = -\cos EAD \cos EDA$$

$$= -\sin FAD \cos EDA$$

$$(\text{because } FAD = EAD + EAF = EAD + 90°).$$

Taking the sum of the squares,

$$\cos^2\beta + \cos^2\gamma = \cos^2 EDA$$

$$= 1 - \sin^2 EDA.$$

Because $EDA = ADB - EDB = 90° - \alpha$,

$$\sin^2 EDA = \cos^2\alpha.$$

Hence

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \quad \text{Q.E.D.} \qquad (4)$$

129. *Corollary.* In the same way that we proved the corollary of the last theorem we may show:

If a plane intersects a system of three rectangular axes, forming with them the respective angles a, b, c, then

$$\sin^2 a + \sin^2 b + \sin^2 c = 1. \tag{5}$$

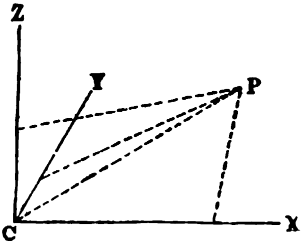
Exercises.

1. Having a system of three rectangular axes OX, OY, OZ, a line OP is drawn from O, making

$$\text{Angle } XOP = 60°;$$
$$\text{Angle } YOP = 50°.$$

Find the angle ZOP and the angles which OP makes with the three planes XOY, YOZ, ZOX.

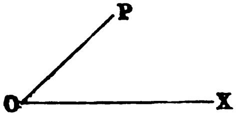
2. Supposing $OP = 32.965$ centimetres, and the angles to have the values in the preceding exercise, find the lengths of the perpendiculars dropped from P upon each of the axes OX, OY, and OZ, and the distances of the feet of these perpendiculars from the origin O.



3. The same thing being supposed, what are the lengths of the respective perpendiculars dropped from P upon the three planes XOY, YOZ, and ZOX?

4. The same thing being supposed, what are the lengths of the projections of OP upon the three planes?

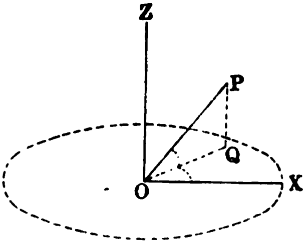
130. *Methods of defining the direction of a line in space.* The direction of a line in a plane is defined by the angle which it makes with some fixed line in the plane. For example, if we have a known fixed line OX, and it is required that another line OP through O shall make an angle $+ 45°$ with OX, this completely fixes the direction of OP. That is, there is only one line through O which makes an angle of $+ 45°$ with OX, when we employ the method of measuring angles defined in Plane Trigonometry, Chap. I.



But if OP is not confined to one plane, its position is not fixed by the angle XOP, because any number of lines may be drawn through O, some above the plane of the paper and some below it, all making the same angle with OX. The student will see that these lines are all elements of a cone having O as its vertex and OX as its axis.

Hence at least two conditions are necessary to define the direction of a line in space. These two quantities may be chosen in various ways, of which the following is the most common.

We have (1) a plane of reference, the position of which we suppose fixed, and (2) in this plane we have a fixed line of reference OX. We call the plane of reference the *fundamental plane.*



Let OZ be a line through O, perpendicular to the plane.

Let OP be the line of which the direction is to be defined.

From any point P of this line drop a perpendicular PQ upon the fundamental plane, and join OQ.

The direction of OP is then defined by the following two angles:

(1) The angle POQ which OP forms with its projection OQ; that is, the angle between OP and the plane.

(2) The angle XOQ which the projection of OP makes with OX.

It will be remarked that the planes of these two angles are perpendicular to each other.

To show that these two angles completely fix the direction of OP, we first remark that when the angle XOQ is given the line OQ is fixed.

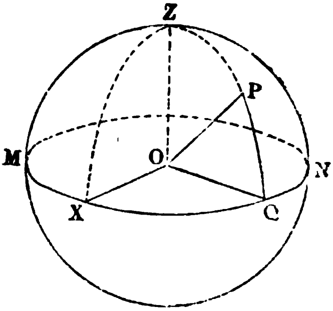
Next, because PQ is perpendicular to the plane, the point P and therefore the line OP must lie in the plane ZOQ, which is fixed because its two lines OZ and OQ are fixed. If the angle QOP in this (vertical) plane is given, there is only one line OP which can form this angle.

Hence *the direction of the line OP is completely determined by the two angles XOQ and QOP.*

The plane XOQ is, when used in this way, the fundamental plane.

131. *Relation of the preceding system to latitude and longitude.* To form another conception of these two angles, pass a sphere around O as a centre, and mark on its surface the points and lines in which the lines and planes belonging to the preceding figure intersect it. Then:

The fundamental plane OXQ intersects the spherical surface in the great circle $MXQN$.



The line OX intersects it in X.

The line OQ intersects it in Q.

The lines OP and OZ intersect it in P and Z.

We therefore have

$$\text{Angle } XOQ = \text{arc } XQ;$$
$$\text{Angle } QOP = \text{arc } QP.$$

If now we imagine this sphere to be the earth, the great circle MN to be its equator, Z to be one of its poles, and P any point on its surface, then—

The arc QP or the angle QOP is the latitude of P.

The arc XQ or angle XOQ is the longitude of P, counted from ZX as a prime meridian.

Thus the angles we have been defining may be described under the familiar forms of longitude and latitude.

132. *Position of a point.* To fix the position of any point relative to a fundamental plane, we must select a point O in that plane and a line OX as a point and line of reference. If P be any point of which we wish to describe the position, we draw the line OP and form the same construction as in article 130. Then the position of P is fixed by the two angles XOQ and QOP, already described, and the distance OP.

Hence *three quantities are required to fix the position of a point in space.*

Def. The quantities which fix or describe the position of a point are called the **co-ordinates** of the point.

The angles XOQ and QOP and the length OP are therefore co-ordinates of the point P, and are distinguished as *polar co-ordinates.*

133. *Polar distance and longitude.* In the preceding figure, because $MXQN$ is a great circle, and Z its pole, we have

$$\text{Angle } XOQ = \text{arc } XQ = \text{angle } XZP;$$
$$\text{Arc } PQ = 90° - \text{arc } ZP.$$

Therefore if we put

p, the arc ZP, which is called the *polar distance* of P;
φ, the arc PQ = angle QOP,

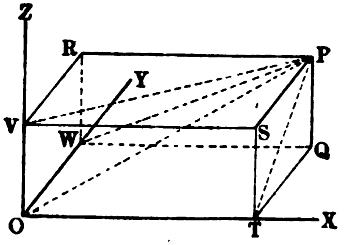
we have

$$\left.\begin{aligned} \sin\varphi &= \cos p; \\ \cos\varphi &= \sin p; \\ \lambda &= \text{angle } XZP. \end{aligned}\right\} \qquad (6)$$

Thus we may define the direction of a line by polar distance and longitude, as well as by latitude and longitude. Applying the same system to positions on the earth's surface, their distance from the north pole of the earth is used instead of their latitude. Thus we should have,

For New York, $p = 49° 17'.3$;
For Rio de Janeiro, $p = 112° 54'.7$.

134. *Rectangular co-ordinates.* The rectangular co-ordinates of a point are its distances from three rectangular planes. In the figure the lines PQ, PR, and PS are the rectangular co-ordinates of P with respect to the axes OZ, OX, and OY respectively. In other words:

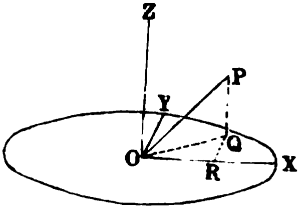


The co-ordinate of a point relative to each axis is the length of the line parallel to that axis from the point to the plane of the two other axes.

We designate these co-ordinates thus: If P is the point whose co-ordinates are x, y and z,

x is the line PR parallel to OX;
y " " PS " " OY;
z " " PQ " " OZ.

135. Problem. *To find the relation between rectangular and polar co-ordinates.*



Let P be the point whose polar co-ordinates are given.

From P drop $PQ \perp$ plane XOY.

From Q drop $QR \perp OX$.

We then have

$$QP = z;$$
$$RQ = y;$$
$$OR = x.$$

The first of these equations follows immediately from the definition of rectangular co-ordinates.

To see the truth of the second we notice that because $PQ \parallel$ plane XOZ, the perpendicular RQ is equal to the perpendicular from P upon the plane XOZ.

Let us now put

φ, the angle QOP, or the elevation of OP, above the plane XOY. We may call this angle the *latitude* of P.

λ, the angle XOQ which OQ, the projection of OP, makes with OX. We may call this angle the *longitude* of P.

r, the length of OP.

We then find

$$\left.\begin{aligned} z &= QP = r \sin \varphi; \\ OQ &= r \cos \varphi; \\ y &= RQ = OQ \sin \lambda = r \cos \varphi \sin \lambda; \\ x &= OR = OQ \cos \lambda = r \cos \varphi \cos \lambda; \end{aligned}\right\} \qquad (7)$$

which are the required expressions.

If we take the sum of the squares of these three equations, we find, by simple reductions,

$$x^2 + y^2 + z^2 = r^2. \qquad (8)$$

136. The second proof of § 126 affords very simple expressions for the rectangular co-ordinates, in terms of the angles which OP makes with the rectangular axes.

In the figure we have

$$\left.\begin{aligned} OT &= x; \\ TQ &= y; \\ QP &= z. \end{aligned}\right\} \qquad (9)$$

Putting, as before, α, β, and γ for the angles POX, POY, and POZ, which OP makes with the axes OX, OY, and OZ respectively, we have, by (b), § 126,

$$\left.\begin{aligned} x &= r \cos \alpha; \\ y &= r \cos \beta; \\ z &= r \cos \gamma. \end{aligned}\right\} \qquad (10)$$

Comparing these with equations (7), we have the following values of α, β, and γ in terms of φ and λ:

$$\left.\begin{aligned} \cos \alpha &= \cos \varphi \cos \lambda; \\ \cos \beta &= \cos \varphi \sin \lambda; \\ \cos \gamma &= \sin \varphi. \end{aligned}\right\} \qquad (11)$$

The sum of the squares of the second members is identically equal to unity, as it should be.

137. We may find a linear expression for r in terms of the rectangular co-ordinates by multiplying the equations (10) by $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ respectively, and taking the sum of the products, noting equation (2). We thus find

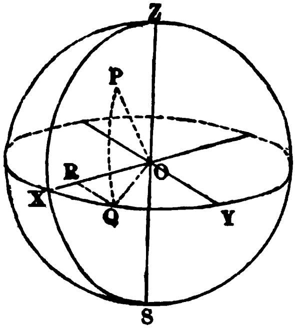
$$x \cos \alpha + y \cos \beta + z \cos \gamma = r.$$

138. The expressions (10) furnish us another definition of the rectangular co-ordinates of a point in space, which will sometimes be useful. Since OTP is a right angle, the point T is the projection of P upon the axis OX. In the same way W and V are the projections of P upon the other axes. Because OT, OW, and OV are the co-ordinates of P, we have the definition:

The rectangular co-ordinates of a point are the distances from the origin to its projections upon the three axes.

139. The preceding formulæ are directly applicable to positions upon the surface of the earth. Let us suppose O to be the centre of the earth; OZ, its axis, Z being the north pole; P, a point on its surface; $OY \perp OX$; the great circle XY the equator, and the plane XOY the plane of the equator.

Also let the great circle ZX be the meridian of Greenwich, from which longitudes are counted, and let PQ be the perpendicular arc of a great circle from P upon the equator.



The three rectangular planes will then be:

Plane XOY, the plane of the earth's equator;

Plane ZOX, the plane passing through the earth's axis and through Greenwich;

Plane YOZ, the plane at right angles to the other two.

The angle $QOP = \varphi$ will then be the latitude of the point P, and $XOQ = \lambda$ will be its longitude.

The rectangular co-ordinates x, y, and z will be the distances of the point P from the respective planes.

Hence if we put

ρ, the radius OP of the earth,

we shall have

$z = \rho \sin \varphi$, distance from plane of equator;

$y = \rho \cos \varphi \sin \lambda$, distance from plane of Greenwich;

$x = \rho \cos \varphi \cos \lambda$, distance from third plane.

EXERCISES.

1. Assuming the latitude of Washington to be $+ 38° 53'.6$, its longitude west from Greenwich $77° 3'.0$, and its distance from the earth's centre to be 6369 kilometres, it is required to compute:

(*a*) Its distance from the plane of the equator;

(*b*) Its distance from the earth's axis;

(*c*) The circumference of the circle which it describes every day in consequence of the earth's rotation on its axis;

(*d*) Its distance from the plane passing through Greenwich and the axis of the earth.

2. What is the length of the straight line through the earth from Washington to Melbourne? The position of Melbourne is:

Latitude, $-37° 38'.7$;
Longitude, $144° 58'.7$ east of Greenwich;
Distance, 6369 kilometres from earth's centre.

Find the angle between the two radii by § 124, and then the distance by § 61.

3. In the preceding problem, what are the lengths of the segments into which the straight line from Washington to Melbourne is divided by the plane of the equator? Begin by finding the distance of each point from the plane of the equator.

4. How far apart are the feet of the respective perpendiculars from Washington and Melbourne upon the plane of the equator? And how far is each foot from the centre of the earth?

140. *Projection of one line upon another.* In geometry and trigonometry the projection of a finite line a upon an indefinite line X is defined as the distance between the feet of the perpendiculars dropped from the ends of a upon X. Hence it has been shown that if we put

p, the projection,
α, the angle which a makes with X,

we shall have

$$p = a \cos \alpha.$$

In demonstrating this proposition we have supposed a and X to intersect, and therefore to lie in one plane.

When they are not in one plane a general definition of the angle between two such lines is necessary.

Def. The angle between two non-intersecting lines is the same as the angle formed by one of the lines and an intersecting line parallel to the other.

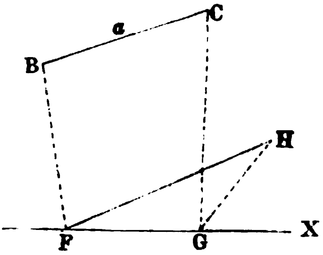
Example. If X and a do not intersect, the angle between X and a is the same as the angle between X and any parallel to a intersecting X.

The theorem which we have proved for a special case is perfectly general, and is as follows:

If α be the angle between any two lines, a the length of one of them, and p the projection of a upon the other, we have

$$p = a \cos \alpha.$$

Proof. Let BC be the projected line; X, the line of projection; BF and CG, the perpendiculars from B and C upon X. (These perpendiculars are not parallel.)



Then FG is by definition the projection of BC upon X.

Through F and G pass planes each perpendicular to the line X.

These planes are parallel (Geom., § 616).

The lines FB and GC being perpendicular to X, lie in these respective planes (Geom., § 586).

Therefore the points B and C lie in these planes.

Through F draw $FH \parallel BC$, and meeting the plane through GC at the point H.

Because the planes and lines are parallel,

$$FH = BC.$$

Join HG. Because H and G are in one plane $\perp$ line X,

$$HG \perp X.$$

Therefore

$$FG = FH \cos HFG;$$

or, because $FH = a$ and angle $HFG = \alpha$,

$$FG = a \cos \alpha. \quad \text{Q.E.D.}$$

141. *Plane triangles in space.* The following problem is of constant occurrence in astronomy:

Given:

(1) The distance and direction of a point B from some point of reference O, the centre of the earth, for example;

(2) The distance and direction of another point C from B;

Required the distance and direction of C from O.

It will be remarked that these distances and directions form the sides of a plane triangle, of which O, B, and C are the vertices.

To attack the problem, we may assume O as the origin of co-ordinates, and find the rectangular co-ordinates of B and C as referred to O.

Let us suppose

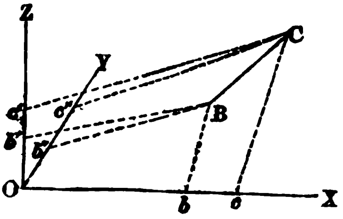
OX, OY, OZ, the three co-ordinate axes;

r, the length OB;*

s, the length BC;

α, β, γ, the angles XOB, YOB, and ZOB, which OB makes with the three co-ordinate axes;

Bb, Bb', Bb'', the perpendiculars from B upon the three axes;



Cc, Cc', Cc'', the perpendiculars from C upon the three axes. Then—

I. Putting x, y, and z for the co-ordinates of B, we have (§ 136)

$$\left.\begin{aligned} x &= Ob = r\cos\alpha; \\ y &= Ob' = r\cos\beta; \\ z &= Ob'' = r\cos\gamma. \end{aligned}\right\} \qquad (12)$$

II. If we designate the angles between BC and the co-ordinate axes by α', β', and γ', we have (§ 140)

$$\left.\begin{aligned} bc &= s\cos\alpha'; \\ b'c' &= s\cos\beta'; \\ b''c'' &= s\cos\gamma'. \end{aligned}\right\} \qquad (13)$$

III. The lines Oc, Oc', Oc'' are by definition the co-ordinates of the point C. Hence if we put $r'' = OC$, and α'', β'', γ'' the angles which OC makes with the three co-ordinate axes, we have

$$\begin{aligned} r''\cos\alpha'' &= Oc; \\ r''\cos\beta'' &= Oc'; \\ r''\cos\gamma'' &= Oc''; \end{aligned}$$

or, comparing with (12) and (13),

$$\left.\begin{aligned} r''\cos\alpha'' &= r\cos\alpha + s\cos\alpha'; \\ r''\cos\beta'' &= r\cos\beta + s\cos\beta'; \\ r''\cos\gamma'' &= r\cos\gamma + s\cos\gamma'. \end{aligned}\right\} \qquad (a)$$

In practice the angles α, β, γ, etc., are not generally convenient, because they are not independent. We therefore substitute for them the polar co-ordinates φ and λ, defined in § 135, putting

---

* The lines OB and OC are omitted in the figure in order to avoid complexity.

φ and λ, the latitude and longitude expressing the direction of OB.

φ', λ', the same for BC. We may conceive φ' to be the angle which CB produced makes with the plane of XY; or, which is the same thing, the angle between BC and a plane through B parallel to XY. We also put

φ'', λ'', the corresponding quantities for OC.

The general equations (a) combined with (11) then give

$$\left.\begin{aligned} r'' \cos\varphi'' \cos\lambda'' &= r\cos\varphi\cos\lambda + s\cos\varphi'\cos\lambda';\\ r''\cos\varphi''\sin\lambda'' &= r\cos\varphi\sin\lambda + s\cos\varphi'\sin\lambda';\\ r''\sin\varphi'' &= r\sin\varphi + s\sin\varphi'. \end{aligned}\right\}\quad(14)$$

When the six quantities which enter into the second members are all given, r'', φ'', and λ'' may be computed by Prob. VI. Chap. IV. But the first two equations may be simplified by the following transformation:

Multiply the first equation by $\sin\lambda'$, and the second by $\cos\lambda'$, and subtract the first product from the second. The remainder reduces to

$$r''\cos\varphi''\sin(\lambda''-\lambda') = r\cos\varphi\sin(\lambda-\lambda').$$

Now multiply the first by $\cos\lambda'$, and the second by $\sin\lambda'$, and add the products. The sum reduces to

$$r''\cos\varphi''\cos(\lambda''-\lambda') = r\cos\varphi\cos(\lambda-\lambda') + s\cos\varphi'.$$

From these two equations and the third of (14) the values of r'', φ'', and $\lambda''-\lambda'$ may be computed by Prob. VI. Chap. V.

We might equally have effected the transformation by using $\sin\lambda$ and $\cos\lambda$ as multipliers, and proceeding exactly as before. The equations to be solved would then be

$$\left.\begin{aligned} r''\cos\varphi''\sin(\lambda''-\lambda) &= s\cos\varphi'\sin(\lambda'-\lambda);\\ r''\cos\varphi''\cos(\lambda''-\lambda) &= r\cos\varphi + s\cos\varphi'\cos(\lambda'-\lambda);\\ r\sin\varphi'' &= r\sin\varphi + s\sin\varphi'. \end{aligned}\right\}\quad(15)$$

NEWCOMB'S MATHEMATICAL COURSE

LOGARITHMIC AND OTHER
MATHEMATICAL TABLES

WITH EXAMPLES OF THEIR USE AND HINTS ON THE ART OF
COMPUTATION

BY

SIMON NEWCOMB

Professor of Mathematics, United States Navy

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PREFACE.

In the present work an attempt is made to present to computers and students a set of logarithmic and trigonometric tables which shall have all the conveniences familiar to those who use German tables. The five-figure tables of F. G. Gauss, of which fifteen editions have been issued, have, after long experience with them, been taken as the basis of the present ones, but modifications have been introduced wherever any improvement could be made.

Five places of decimals have been adopted as an advantageous mean. The results obtained by them, being nearly always reliable to the 10,000th part, are amply accurate for most computations, while the time of the student who uses them is not wasted in unnecessary calculation.

The Introduction is intended to serve not only as an explanation of the tables, but as a little treatise on the art of computation, and the methods by which the labor of computation may be abridged.

To avoid fostering the growing evil of nearsightedness among students, the author and publishers have spared neither pains nor expense in securing clearness of typography.

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TABLE I.

LOGARITHMS OF NUMBERS.

1. Introductory Definitions.

Natural numbers are numbers used to represent quantities.

The numbers used in arithmetic and in the daily transactions of life are natural numbers.

To every natural number may be assigned a certain other number, called its **logarithm.**

The **logarithm** of a natural number is the exponent of the power to which some assumed number must be raised to produce the first number. The assumed number is called the **base.** *E.g.*, the logarithm of 100 with the base 10 is 2, because $10^2 = 100$; with the base 2, the logarithm of 64 would be 6, because $2^6 = 64$.

A **system of logarithms** means the logarithms of all positive numbers to a given base.

Although there may be any number of systems of logarithms, only two are used in practice, namely:

1. The natural or Napierian system, base $= e = 2.718\,282$.
2. The common system, base $= 10$.

The natural system is used for purely algebraic purposes.

The common system is used to facilitate numerical calculations and is the only one employed in this book.

If the natural number is represented by n, its logarithm is called

$$\log n.$$

A logarithm usually consists of an integer number and a decimal part.

The integer is called the **characteristic** of the logarithm.

The decimal part is called the **mantissa** of the logarithm.

A **table of logarithms** is a table by which the logarithm of any given number, or the number corresponding to any given logarithm, may be found.

The most simple form of table is that on the first page of Table I., which gives the logarithms of all entire numbers from 1 to 150; each logarithm being found alongside its number. The student may begin his exercises with this table.

Mathematical tables in general enable us, when one of two related quantities is given, to find the other.

In such tables the quantity supposed to be given is called the **argument.**

The argument is usually printed on the top, bottom, or side of the table.

The quantities to be found are called **functions** of the argument, and are found in the same columns or lines as the argument, but in the body of the table.

In a table of logarithms the natural number is the argument, and the logarithm is the function.

2. The Use of Logarithms.

The following properties of logarithms are demonstrated in treatises on algebra.

I. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

II. *The logarithm of a quotient is found by subtracting the logarithm of the divisor from that of the dividend.*

III. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

IV. *The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

We thus derive the following rules:

To find the product of several factors by logarithms.

RULE. *Add the logarithms of the several factors. Enter the table with the sum as a new logarithm, and find the number corresponding to it.*

This number is the product required.

Example 1. To multiply 7 × 8.

We find from the first page of Table I.

$$\begin{aligned} \log 7 &= 0.845\,10 \\ ``\ \ 8 &= 0.903\,09 \\ \hline \text{Sum of logs} &= 1.748\,19 = \text{log of product.} \end{aligned}$$

Having added the logarithms, we look in column log for a number corresponding to 1.788 19 and find it to be 56, which is the product required.

Ex. 2 To find the continued product 2 × 6 × 8.

log 2, 0.301 03
" 6, 0.778 15
" 8, 0.903 09

Sum of logs, 1.982 27 = log product.

The number corresponding to this logarithm is found to be 96, which is the product required.

Ex. 3. To find the quotient of 147 ÷ 21.

log 147, 2.167 32
" 21, 1.322 22

Difference, 0.845 10

We find this difference to be the logarithm of 7, which is the required quotient.

Ex. 4. To find the quotient arising from dividing the continued product 98 × 102 × 148 by the continued product 21 × 37 × 68.

log 21, 1.322 22	log 98, 1.991 23
" 37, 1.568 20	" 102, 2.008 60
" 68, 1.832 51	" 148, 2.170 26
Sum = log divisor, 4.722 93	Sum = log dividend, 6.170 09
	log divisor, 4.722 93
	Difference = log quotient, 1.447 16

Looking into the table, we find the number corresponding to this logarithm to be 28, which is the required quotient.

NOTE. The student will notice that we have found this quotient without actually determining either the divisor or dividend, having used only their logarithms. If he will solve the problem arithmetically, he will see how much shorter is the logarithmic process.

Ex. 5. To find the seventh power of 2.

We have log 2 = 0.301 03
7

2.107 21 = log 128

Hence 128 is the required power.

Ex. 6. To find the cube root of 125.

3 | 2.096 91
0.698 97

The index of the root being 3, we divide the logarithm of 125 by it. Looking in the tables, we find the number to be 5, which is the root required.

EXERCISES.

Compute the following products, quotients, powers, and roots by logarithms.

1. $11 . 13$. Ans. 143.

2. 12^2. Ans. 144.

3. $\frac{12^3}{6^2}$. Ans. 48.

4. $\frac{2 . 9^2 . 91 . 78}{13^2 . 21 . 3}$. Ans. 108.

5. $\frac{22 . 8^2}{\sqrt{121}}$. Ans. 128.

6. $\frac{51 . 98 \sqrt{81}}{34 . 63}$. Ans. 21.

7. $\frac{2^7 . 3^5}{6^3}$. Ans. 144.

8. $\frac{54 . 48}{8 . 9}$. Ans. 36.

3. Arrangement of the Table of Logarithms.

A table giving every logarithm alongside its number, as on the first page of Table I., would be of inconvenient bulk. For numbers larger than 150 the succeeding parts of Table I. are therefore used. Here the first three figures of the natural number are given in the left-hand column of the table. The first figure must be understood where it is not printed. The fourth figure is to be sought in the horizontal line at the top or bottom. The mantissa of the logarithm is then found in the same line with the first three digits, and in the column having the fourth digit at the top.

To save space the whole logarithm is not given in the column, but only its last three figures. The first two figures are found in the first column, and are commonly the same for all the logarithms in any one line.

Example 1. To find the logarithm of 2090.

We find the number 209, the figure 2 being omitted in printing, in the left-hand column of the table, and look in the column having the fourth figure, 0, at its top or bottom. In this column we find 320 15, which is the mantissa of the logarithm required.

Ex. 2. To find the logarithm of 2092.

Entering the table with 209 in the left-hand column, and choosing the column with 2 at the top, we find the figures 056. To these we prefix the figures 32 in column 0, making the total logarithm .320 56. Therefore

Mantissa of log 2092 = .320 56.

EXERCISES.

Find in the same way the mantissæ of the logarithms of the following numbers:

2240;
2242;
2249;
2895;
3644;
4688;
5133;
5256;
5504;
8925;
9557;
9780.

When the first two figures of the mantissa are not found in the same line in which the number is sought, they are to be found in the first line above which contains them.

Example. The first two figures of log 6250 are 79, which belongs to all the logarithms below as far as 6309. Therefore mantissa of log 6250 = .795 88.

EXERCISES.

Find the mantissæ of the logarithms of

6300; answer, .799 34.
6309; " .799 96.
6434;
6653;
6755;
6918;
7868.

Exception. There are some cases in which the first two figures change in the course of the line. In this case the first two figures are to be sought in the line above before the change and in the line next below after the change.

Example. The mantissa of log 6760 is .829 95. But the mantissa of log 6761 is .830 01. In this case the figures 83 are to be found in the next line below. To apprise the computer of these cases, each of the logarithms in which the two first figures are found in the line below is indicated by an asterisk.

EXERCISES.

Find the mantissa of

log 1022; answer, .009 45.
log 1024; " .010 30.

1231;
1387;
1419;
1621;
1622;
1862;
1863;
1999;
3988;
4675;
4798;
5377;
8512;
1009.

4. Characteristics of Logarithms.

The part of the table here described gives only the *mantissa* of each logarithm. The characteristic must be found by the general theory of logarithms.

The following propositions are explained in treatises on algebra:

The logarithm of 1 is 0.
" " " 10 " 1.
" " " 100 " 2.
" " " 1000 " 3.
" " " 10^n " n.

Since any number of one digit is between 0 and 10, its logarithm is between 0 and 1; that is, it is 0 *plus* some fraction. In the same way, the logarithm of a number of two digits is 1 + a fraction. And in general,

The characteristic of the logarithm of any number greater than 1 *is less by unity than the number of its digits preceding the decimal point.*

Example. The characteristic of the logarithm of any number between 1 and 10 is 0; between 10 and 100 it is 1; between 100 and 1000 it is 2, etc.

Characteristic of log 1646 is 3.
" " " 164.6 " 2.
" " " 16.46 " 1.
" " " 1.646 " 0.

It is also shown in algebra that if a number be divided by 10 we diminish its logarithm by unity.

Logarithms of numbers less than unity are most conveniently expressed by making the characteristic alone negative.

For example:

$$\log 0.2 = \log 2 - 1 = -1 + .301\,03;$$

$$\text{“}\ \ 0.02 = \log 2 - 2 = -2 + .301\,03.$$

Hence: *The mantissæ of the logarithms of all numbers which differ only in the position of the decimal point are the same.*

Hence, also, in seeking a logarithm from the table we find the mantissa without any reference to the decimal point. Afterward we affix the characteristic according to the position of the decimal point.

For convenience, when a negative characteristic is written the minus sign is put above it to indicate that it extends only to the characteristic below it and not to the mantissa. Thus we write

$$\log .02 = \bar{2}.301\,03.$$

In practice, however, it is more common to avoid the use of negative characteristics by increasing them by 10. We then write

$$\log .02 = 8.301\,03 - 10.$$

If we omitted to write -10 after the logarithm, the latter would, in strictness, be the log of 2×10^8. But numbers so great as this product occur so rarely in practice that it is not generally necessary to write -10 after the logarithm. This may be understood.

A convenient rule for remembering what characteristic belongs to the logarithm of a decimal fraction is:

The characteristic is equal to 9, minus the number of zeros after the decimal point and before the first significant figure.

Examples. log 34060 = 4.532 24
" 340.60 = 2.532 24
" 3.4060 = 0.532 24
" .034 06 = 8.532 24 − 10
" .000 340 6 = 6.532 24 − 10

It will be seen that we can find the logarithms of numbers from 1 to 150 without using the first page of the table at all, since all the mantissæ on this page are found on the following pages as logarithms of larger numbers.

EXERCISES.

Find the logarithms of the following numbers:

1.515 .003 899
.01 702 0.4276
18.62 464 700
.03 735 98.030

Find the numbers corresponding to the following logarithms:

3.241 80; 8.750 35 − 10; 9.999 91 − 10;
1.191 45; 7.411 28 − 10; 5.999 96;
5.653 21; ans. 450 000 6.889 97 − 10; 2.960 28;
6.748 27; ans. 5 601 000 9.116 94 − 10; 0.886 27;
7.560 03; ans. 36 310 000 7.250 18 0.000 87.

5. Interpolation of Logarithms.

In all that precedes we have used only logarithms of numbers containing not more than 4 significant digits. But in practice numbers of more than four figures have to be used. To find the logarithms of such numbers the process of interpolation is necessary. This process is one of simple proportion, which can be seen from the following example.

To find log. 1167.23.

The table gives the logarithms of 1167 and of 1168, which we find to be as follows:

log 1167 = 3.067 07
" 1168 = 3.067 44
Difference of logarithms = .000 37

Now the number of which we wish to find the logarithm being between these numbers, its logarithm is between these logarithms; that is, it is equal to 3.067 07 *plus* a fraction less than .000 37.

Since the difference 37 corresponds to the difference of unity in the two numbers, we assume that the quantity to be added to the logarithm bears the same proportion to .23 that 37 does to unity. We therefore state the proportion

1 : .23 :: 37 : increase required.

The solution of this proportion gives $.23 \times 37 = 8.51$, which is the quantity to be added to log 1167 to produce the logarithm required.* The result is 3.067 155 1.

But our logarithms extend only to five places of decimals, while the result we have written has seven. We therefore take only five places of decimals. If we write the mantissa 3.067 15, the result will be too small by .51. If we write 3.067 16, it will be too great by .49. Since the last result is nearer than the first, we give it the preference, and write for the required logarithm

log 1167.23 = 3.067 16.

We thus have the following rule for interpolating:

Take from the table the logarithm corresponding to the first four significant digits of the number.

Considering the following digits as a decimal fraction, multiply the difference between the logarithm and the next one following by such decimal fraction.

---

* In this multiplication we have used a decimal point to mark off the fifth order of decimals. This is a convenient process in all such computations.

This product being added to the logarithm of the table will give the logarithm required.

The whole operation by which we have found log 1167.23 would then be as follows:

$$\begin{array}{rr}
\log 1167 = 3.067\,07 & \\
37 \times 0.2 \qquad\qquad & 7.4 \\
\times 0.03 \qquad\qquad & 1.11 \\
\hline
\log 1167.23 = 3.067\,16 &
\end{array}$$

The products for interpolation, 7.4 and 1.11, may be found by multiplying by the fifth and sixth figures of the number separately.

To facilitate this multiplication, tables of proportional parts are given in the margin. Each difference between two logarithms will be readily found in heavy type not far from that part of the table which is entered, and under it is given its product by .1, .2, etc., . . .9. We therefore enter this little table with the fifth figure, and take out the corresponding number to be added to the logarithm. Then if there is a sixth figure, we enter with that also and move the decimal one place to the left. We then add the two sums to the logarithm.

6. Labor-saving Devices.

In using a table of logarithms, the student should accustom himself to certain devices by which the work may be greatly facilitated.

In the first place it is not necessary to take the whole difference between two consecutive logarithms. He has only to subtract the last figure of the preceding logarithm from the last one of the following, increased by 10 if necessary, and thus find the last figure of the difference.

The nearest difference in the margin of the table having this same last figure will always be the difference required.

Example. If the first four figures of the number are 1494, instead of subtracting 435 from 464 we say 5 from 14 leaves 9, and look for the nearest difference which has 9 for its last figure. This we readily find to be 29, at the top of the next page.

NOTE. In nearly all cases the difference will be found on the same page with the logarithm. The only exception is at the bottom of the first page, where, owing to the number of differences, they cannot all be printed.

In the preceding examples we have written down the numbers in full, which it is well that the beginner should do for himself. But after a little practice it will be unnecessary to write down anything but the logarithm finally taken out. The student should accustom himself to take the proportional parts mentally, adding them to the logarithm of the table and writing down the sum at sight. The habit of doing this easily and correctly can be readily acquired by practice.

Exercises. Find the logarithms of

792 638;
1000.77;
1000.07;
100 007;
181 982;
281.936;
0.99997;
949.916;
20.8962;
660 652;
77.642;
8.8953.

As a precaution in taking out logarithms, the computer should always, after he has got his result, look into the table and see that it does really fall between two consecutive logarithms in the table.

If the fraction to be interpolated is nearly unity, especially if it is equal to or greater than 9, it will generally be more convenient to multiply the difference of the logarithms by the complement* of the fraction and subtract the product from the logarithm next succeeding. The following are examples of the two methods, which may always be applied whether the fraction be large or small:

Example 1. log 1004.28 = log (1005 — .72).

log 1004,		.001 73	log 1005,		.002 17
pr. pt. for	.2,	8.8	pr. pt. for	.7,	— 30.8
" " "	.08,	3.5	" " "	.02,	— .9
	log,	3.001 85		log,	3.001 85

Ex. 2. log 154 993 = 155 000 — 7.

log 1549,		.190 05	1550,		.190 33
pr. pt. for	.9,	25.2	pr. pt. for	—.07,	—1.96
" " "	.03,	0.8		log,	5.190 31
	log,	5.190 31			

---

* By the *complement* or *arithmetical complement* of a decimal fraction is here meant the remainder found by subtracting it from unity or from a unit of the next order higher than itself. Thus:

co. .723 = .327
co. .1796 = .8204
co. .9932 = .0068.

7. To find the Number corresponding to a given Logarithm.

The reverse process of finding the number corresponding to a given logarithm will be seen by the following example:

To find the number of which the logarithm is 2.027 90.

Entering the table, we find that this logarithm does not exactly occur in the table. We therefore take the next smaller logarithm, which we find to be as follows:

$$\log 1066 = 2.027\,76.$$

Subtracting this from the given logarithm we find the latter to be greater by 14, while the difference between the two logarithms of the table is 40. We therefore state the proportion

40 : 14 :: 1 to the required fraction.

The result is obtained by dividing 14 by 40, giving a quotient .35. The required number is therefore 106.635. It will be remarked that we take no account of the characteristic and position of the decimal until we write down the final result, when we place the decimal in the proper position.

The table of proportional parts is used to find the fifth and sixth figures of the number by the following rule:

If the given logarithm is not found in the table, note the excess of the given logarithm above the next smaller one in the table, which call Δ.

Take the difference of the two tabular logarithms, and find it among the large figures which head the proportional parts.

That proportional part next smaller than Δ will be the fifth figure of the required number.

Take the excess of Δ above this proportional part; imagine its decimal point removed one place to the right, and find the nearest number of the table.

This number will be the sixth figure of the required number.

Example. To find the number of which the logarithm is 2.193 59.

Entering the table, we find the next smaller logarithm to be .193 40. Therefore $\Delta = 19$.

Also its tabular difference $= 28$.

Entering the table of proportional parts under 28, we find 16.8 opposite 6 to be the number next smaller than 19 the value of Δ. Therefore the fifth figure of the number is 6.

The excess of 19 above 16.8 is 2.2. Looking in the same table for the number 22, we find the nearest to be opposite 8.

Therefore the fifth and sixth figures of the required number are 68. Now looking at the log .193 40 and taking the corresponding number, we find the whole required number to be

156 168.

The characteristic being 2, the number should have three figures before the decimal point. Therefore we insert the decimal point at the proper place, giving as the final result 156.168.

8. Number of Decimals necessary.

In the preceding examples we have shown how with these tables the numbers may be taken out to six figures. In reality, however, it will seldom be worth while to write down more than five figures. That is, we may be satisfied by adding only one figure to the four found from the table. In this case, when we enter the table of proportional parts, we take only the number corresponding to the nearest proportional part.

To return to the last preceding example, where we find the number corresponding to 2.193 59. We find under the difference 28 that the number nearest 19 is 19.6, which is opposite 7.

Therefore the number to be written down would be 156.17.

In the following exercises it would be well for the student to write six figures when the number is found on one of the first two pages of the table and only five when on one of the following pages. The reason of this will be shown subsequently.

EXAMPLES AND EXERCISES.

1. To find the square root of $\frac{3}{2}$.

We have log 3, 0.477 12

" 2, 0.301 03

log $\frac{3}{2}$, 0.176 09

÷ 2, log $\sqrt{\frac{3}{2}}$, 0.088 04

Here we have a case in which the half of an odd number is required. We might have written the last logarithm 0.088 045, but we should then have had six decimals, whereas, as our tables only give five decimals, we drop the sixth. If we write 4 for the fifth figure it will be too small by half a unit, and if we write 5 it will be too large by half a unit. It is therefore indifferent which figure we write, so far as mere accuracy is concerned.

A good rule to adopt in such a case is to ***write the nearest*** **EVEN** ***number.*** For example,

for the half of .261 81 we write .130 90;
" " .261 83 " .130 92;
" " .261 85 " .130 92;
" " .261 87 " .130 94;
" " .261 89 " .130 94;
" " .261 97 " .130 98;
" " .261 99 " .131 00.

Returning to our example, we find, by taking the number corresponding to 0.088 04,

$$\sqrt{\tfrac{3}{2}} = 1.224\ 72.$$

2. To find the square root of $\frac{2}{3}$.

$$\begin{array}{ll} \log 2, & 0.301\ 03 \\ \text{"}\ \ 3, & \underline{0.477\ 12} \\ \log \tfrac{2}{3}, & 9.823\ 91 - 10 \\ \tfrac{1}{2}\log \tfrac{2}{3}, & 4.911\ 96 - 5 = \log \sqrt{\tfrac{2}{3}}. \end{array}$$

The last logarithm is the same as

$$9.911\ 96 - 10,$$

which is the form in which it is to be written in order to apply the rule of characteristics. The corresponding number is 0.816 50.

We have here a case in which, had we neglected considering the surplus − 10 as we habitually do, the characteristic of the answer would have been 4 instead of 9 or − 1. The easiest way to treat such cases is this:

When we have to divide a logarithm in order to extract a root, instead of increasing the characteristic by 10, *increase it by* 10 × *index of root.*

Thus we write $\log \frac{2}{3} = 19.823\ 91 - 20.$

Dividing by 2, $\log \sqrt{\frac{2}{3}} = 9.911\ 96 - 10,$

which is in the usual form.

3. To find the cube root of $\frac{1}{2}$.

$$\begin{array}{ll} \log 1, & 0.000\ 00 \\ \text{"}\ \ 2, & \underline{0.301\ 03} \\ \log \tfrac{1}{2}, & 9.698\ 97 - 10, \end{array}$$

which we write in the form

$$\log \tfrac{1}{2} = 29.698\ 97 - 30.$$

Dividing this by 3,

$$\tfrac{1}{3}\log \tfrac{1}{2} = \log \sqrt[3]{\tfrac{1}{2}} = 9.899\ 66 - 10.$$

This logarithm is in the usual form, and gives

$$\sqrt[3]{\tfrac{1}{2}} = 0.793\,70.$$

The affix -30, or $-10 \times$ divisor, can be left to be understood in these cases as in others. All that is necessary to attend to is that instead of supposing the characteristic to be one or more units less than 10, as in the usual run of cases, we suppose it to be one or more units less than $10 \times$ divisor.

Find:

4. The square root of $\frac{1}{2}$;
5. The cube root of 2;
6. The fourth root of $\frac{3}{4}$;
7. The fifth root of 20;
8. The tenth root of 10;
9. The tenth root of $\frac{1}{10}$.

9. The Arithmetical Complement.

When a logarithm is subtracted from zero, the remainder is called its *arithmetical complement.*

If L be any logarithm, its arithmetical complement will be $-L$. Hence if

$$L = \log n,$$

then

$$\text{arith. comp.} = -L = \log \frac{1}{n};$$

that is,

The arithmetical complement of a given logarithm is the logarithm of the reciprocal of the number corresponding to the given logarithm.

Notation. The arithmetical complement of a logarithm is written co-log. It is therefore defined by the form

$$\text{co-log}\, n = \log \frac{1}{n}.$$

Finding the arithmetical complement. To find the arithmetical complement of $\log 2 = 0.301\,03$, we may proceed thus:

$$\begin{array}{rl} & 0.000\,00 \\ \log 2, & \underline{0.301\,03} \\ \text{co-log}\ 2, & 9.698\,97 - 10. \end{array}$$

We subtract from zero in the usual way; but when we come to the characteristic, we subtract it from 10. This makes the remainder too large by 10, so we write -10 after it, thus getting a quantity which we see to be $\log 0.5$.

We may leave the -10 to be understood, as already explained.

The arithmetical complement may be formed by the following rule:

Subtract each figure of the logarithm from 9, *except the last significant one, which subtract from* 10. *The remainders will form the arithmetical complement.*

For example, having, as above, the logarithm 0.301 03, we form, mentally, $9 - 0 = 9$; $9 - 3 = 6$; $9 - 0 = 9$; $9 - 1 = 8$; $9 - 0 = 9$; $10 - 3 = 7$; and so write

9.698 97

as the arithmetical complement.

To form the arithmetical complement of 3.284 00 we have $9 - 3 = 6$; $9 - 2 = 7$; $9 - 8 = 1$; $10 - 4 = 6$. The complement is therefore

.716 00.

The computer should be able to form and write down the arithmetical complement without first writing the tabular logarithm, the subtraction of each figure being performed mentally.

Use of the arithmetical complement. The co-log is used to substitute addition for subtraction in certain cases, on the principle: *To add the co-logarithm is the same as to subtract the logarithm.*

Example. We may form the logarithm of $\frac{3}{2}$ in this way by addition:

$$\begin{array}{rr} \log 3, & 0.477\,12 \\ \text{co-log } 2, & \underline{9.698\,97} \\ \log \frac{3}{2}, & 0.176\,09 \end{array}$$

Here there is really no advantage in using the co-log. But there is an advantage in the following example:

To find the value of $P = \frac{2763 \times 419.24}{99}$. We add to the logarithms of the numerator the co-log of the denominator, thus:

$$\begin{array}{lrr} \text{og} & 2763, & 3.441\,38 \\ \log & 419.24, & 2.622\,46 \\ \text{co-log} & 99, & \underline{8.004\,36} \\ \log P, & & 4.068\,20 \end{array}$$

$$\therefore P = 11\,700.$$

The use of the arithmetical complement is most convenient when the divisor is a little less than some power of 10.

EXERCISES.

Form by arithmetical complements the values of:

1. $\dfrac{109 \times 216.26}{0.99316}$

2. $\dfrac{8263 \times 9162.7}{92 \times 99.618}$

3. $\dfrac{4 \times 6 \times 8219}{9 \times 992}$

10. Practical Hints on the Art of Computation.

The student who desires to be really expert in computation should learn to reduce his written work to the lowest limit, and to perform as many of the operations as possible mentally. We have already described the process of taking a logarithm from the table without written computation, and now present some exercises which will facilitate this process.

1. *Adding and subtracting from left to right.* If one has but two numbers to add it will be found, after practice, more easy and natural to write the sum from the left than from the right. The method is as follows:

In adding each figure, notice, before writing the sum, whether the sum of the figures following is less or greater than 9, or equal to it.

If the sum is less than 9, write down the sum found, or its last figure without change.

If greater than 9, increase the figure by 1 before writing it down.

If equal to 9, the increase should be made or not made according as the first sum following which differs from 9 is greater or less than 9.

If the first sum which differs from 9 exceeds it, not only must we increase the number by 1, but must write zeros under all the places where the 9's occur. If the first sum different from 9 is less than 9, write down the 9's without change.

The following example illustrates the process:

$$\begin{array}{cccccccccccccccccccc}
 & 7 & 5 & 0 & 2 & 7 & 6 & 8 & 3 & 5 & 7 & 8 & 5 & 8 & 8 & 9 & 2 & 8 & 3 & 7 \\
 & 8 & 2 & 3 & 9 & 1 & 7 & 1 & 6 & 4 & 5 & 0 & 4 & 1 & 1 & 0 & 2 & 5 & 9 & 8 \\
\hline
1 & 5 & 7 & 4 & 1 & 9 & 4 & 0 & 0 & 0 & 2 & 8 & 9 & 9 & 9 & 9 & 5 & 4 & 3 & 5
\end{array}$$

Here 7 and 8 are 15. 5 + 2 being less than 9, we write 15 without change. 3 + 0 being less than 9, we write 7 without change. 9 + 2 being greater than 9, we increase the sum 3 + 0 by 1 and write down 4. 7 + 1 being less than 9, we write the last figure of $9 + 2$, or 1, without change. $6 + 7$ being greater than 9, we increase $7 + 1$ by 1 and write down 9. Under $6 + 7$ we write down 3 or 4. To find which, $8 + 1 = 9$; $3 + 6 = 9$; $5 + 4 = 9$; $7 + 5 = 12$. This first sum which is different from 9 being greater than 9, we write 4 under $6 + 7$, and 0's in the three following places where the sums are 9. $7 + 5 = 12$. Since $8 + 0 < 9$, we write down 2. Before deciding whether to put 8 or 9 under $8 + 0$, we add $5 + 4 = 9$; $8 + 1 = 9$; $8 + 1 = 9$; $9 + 0 = 9$; $2 + 2 = 4$. This being less than 9, we write 8 under $8 + 0$, and 9's in the four following places. Since $5 + 8 = 13 > 9$, we write 5 under $2 + 2$. Since $9 + 3 = 12 > 9$, we write 4 under $5 + 8$. Since $8 + 7 = 15 > 9$, we write 3 under $9 + 3$. Finally, under $8 + 7$ we write 5.

This process cannot be advantageously applied when more than two numbers are to be added.

EXERCISES.

Let the student practise adding each consecutive pair of the following lines, which are spaced so that he can place the upper margin of a sheet of paper under the lines he is adding and write the sum upon it.

2	5	0	9	1	7	2	8	5	3	1	6	9	8	1	2	0	8
2	5	1	2	3	5	9	6	4	6	9	2	1	8	4	3	6	8
7	9	1	6	1	5	8	3	2	3	1	6	6	4	6	8	9	1
2	0	8	5	3	2	1	6	4	3	7	9	1	0	2	9	0	9
8	6	8	5	8	8	9	6	4	3	4	2	9	4	4	8	2	5
9	8	7	6	5	4	3	2	1	0	1	2	3	4	5	6	7	4

Subtracting. We subtract each figure of the subtrahend from the corresponding one of the minuend (the latter increased by 10 if necessary), as in arithmetic.

Before writing down the difference, we note whether the following figure of the subtrahend is greater, less, or equal to the corresponding figure of the minuend.

If greater, we diminish the remainder by 1 and write it down.*

If less, we write the remainder without change.

If equal, we note whether the subtrahend is greater or less than the minuend in the first following figure in which they differ.

If greater, we diminish the remainder by 1, as before, and write 9's under the equal figures.

---

* If the student is accustomed to carrying 1 to the figures of the minuend when he has increased the figure of his subtrahend by 10, he may find it easier to defer each subtraction until he sees whether the remainder is or is not to be diminished by 1, and, in the latter case, to increase the minuend by 1 before subtracting.

If less, write the remainder unchanged, putting 0's under the equal figures.

Example.

$$\begin{array}{cccccccccccccc}
7 & 2 & 2 & 9 & 3 & 5 & 1 & 6 & 2 & 1 & 4 & 3 & 9 & 4 \\
2 & 4 & 2 & 6 & 8 & 5 & 1 & 8 & 0 & 1 & 4 & 1 & 9 & 8 \\
\hline
4 & 8 & 0 & 2 & 4 & 9 & 9 & 8 & 2 & 0 & 0 & 1 & 9 & 6
\end{array}$$

Here $7 - 2 = 5$; because $4 > 2$, we write 4. $12 - 4 = 8$; because $2 = 2$ and $6 < 9$, we write 8; and write 0 in the following place. $9 - 6 = 3$; because $8 > 3$, we write 2. $13 - 8 = 5$; $5 = 5$; $1 = 1$; $8 > 6$; so under $13 - 8$ we write 4, with 9's in the two next places. $16 - 8 = 8$; because $0 < 2$, we write 8. $2 - 0 = 2$; $1 = 1$; $4 = 4$; $1 < 3$; so under $2 - 0$ we write 2, followed by 0's. $3 - 1 = 2$; because $9 = 9$, $8 > 4$, we write 1, with 9 in the next place. $14 - 8 = 6$, which we write as the last figure.

EXERCISES.

The preceding exercises in addition will serve as exercises in subtraction by subtracting each line from that above or below it. The student should be able to subtract with equal facility whether the minuend is written above or below the subtrahend.

Mental addition and subtraction. When an expert computer has to add or subtract two logarithms, as in forming a product or quotient of two quantities, he does not necessarily write both of them, but prefers to write the first and, taking the other mentally, add (or subtract) each figure in order from left to right, and write down the sum (or difference). He thus saves the time spent in writing one number, and, sometimes, the inconvenience of writing it where there is not sufficient room for it.

This process of inverted addition is most useful in adding the proportional part in taking a logarithm from the table. It is then absolutely necessary to save the computer the trouble of copying both logarithm and proportional part.

Expert computers can add seven-figure logarithms in this way without trouble. But with those who do not desire to become experts it will be sufficient to learn to add two or three figures, so as to be able to take a five-figure or seven-figure logarithm from the table without writing anything but the result.

11. Imperfections of Logarithmic Calculations.

Nearly all practical computations with logarithms are affected by certain sources of error, arising from the omission of decimals. It is important that these errors should be understood in order not only to avoid them so far as possible, but to avoid spending labor in aiming at a degree of accuracy beyond that of which the numbers admit.

Mathematical results may in general be divided into two classes: (1) those which are absolutely exact, and (2) those which are only to a greater or less degree approximate.

As an example of the former case, we have all operations upon entire numbers which involve only multiplication and division. For example, the equations

$$16^2 = 256$$

$$\frac{8^2}{6^2} = \frac{16}{9}$$

are absolutely exact.

But if we express the fraction $\frac{1}{7}$ as a decimal fraction, we have

$$\tfrac{1}{7} = .142\,857\ldots, \text{etc.}, \textit{ad infinitum}.$$

Hence the representation of $\frac{1}{7}$ as a decimal fraction can never be absolutely exact. The amount of the error will depend upon how many decimals we include. If we use only two decimals we shall certainly be within one hundredth; if three, within one thousandth, etc. Hence the degree of accuracy to which we attain depends upon the number of decimals employed. By increasing the number of decimals we can attain to any degree of accuracy. As an example, it is shown in geometry that if the ratio of the circumference of a circle to its diameter be written to 35 places of decimals, the result will give the whole circumference of the visible universe without an error as great as the minutest length visible in the most powerful microscope.

There are no numbers, except the entire powers of 10, of which the logarithms can be exactly expressed in decimals. We must therefore omit all figures of the decimal beyond a certain limit. The number of decimals to be used in any case depends upon the degree of accuracy which is required. The large tables of logarithms contain seven decimal places, and therefore give results correct to the ten-millionth part of the unit. This is sufficiently near the truth in nearly all the applications of logarithms.

With five places of decimals our numbers will be correct to the hundred-thousandth part of a unit. This is sufficiently near for most practical applications.

Accumulation of errors. When a long computation is to be made, the small errors are liable to accumulate so that we cannot rely upon this degree of accuracy in the final result. The manner in which the tables are arranged so as to reduce the error to a minimum may be shown as follows:

We have to seven places of decimals

$$\log 17 = 1.230\,448\,9$$
$$\text{“} \quad 18 = 1.255\,272\,5$$

When the tables give only five places of decimals the two last figures must be omitted. If the tables gave log 17=.230 44, the logarithm would be too small by 89 units in the seventh place. It is therefore increased by a unit in the fifth place, and given .230 45. This quantity is then too small by 11, and is therefore nearer the truth than the other. The nearest number being always given, we have the result:

Every logarithm in the table differs from the truth by not more than one half a unit of the last place of decimals.

Since the error may range anywhere from zero to half a unit, and is as likely to have one value as another between those limits, we conclude:

The average error of the logarithms in the tables is one fourth of a unit of the last place of decimals.

Errors in interpolation. When we interpolate the logarithm we add to the tabular logarithm another quantity, the proportional part, which may also be in error by half a unit, but of which the average error will only be one fourth of a unit.

As most logarithms have to be interpolated, the general result will be:

An interpolated logarithm may possibly be in error by a unit in the last place of decimals.

The sum of the average errors will, however, be only half a unit. But these errors may cancel each other, one being too large and the other too small. The theory of probabilities shows that, in consequence of this probable cancellation of errors, the average error only increases as the square root of the number of erroneous units added.

The square root of 2 is 1.41.

If, therefore, we add two quantities each affected with a probable error ± .25, the result will be, for the probable error of the sum,

$$1.41 \times .25 = 0.35.$$

We therefore conclude:

The average error of a logarithm derived from the table by interpolation is 0.35 *of a unit of the last place.*

Applying the above rule of the square root to the case in which several logarithms are added or subtracted to form a quotient, we find the results of the following table:

No. of logs added or subtracted.	Average error.
1	0.35
2	0.50
3	0.63
4	0.72
5	0.81
6	0.88
7	0.95
8	1.02
9	1.08
10	1.14

From this table we see that if we form the continued product of eight factors, by adding their logarithms the average error of the sum of the logarithms will be more than a unit in the last place.

As an example of the accumulation of errors, let us form the product 11 . 13.

We have from the table

$$\begin{aligned} \log 11 &= 1.041\,39 \\ \text{“}\ 13 &= \underline{1.113\,94} \\ \text{log product, } & \ 2.155\,33 \end{aligned}$$

We see that this is less than the given logarithm of the product 143 by a unit of the fifth order. But if we use seven decimals we have

$$\begin{aligned} \log 11,\ & 1.041\,392\,7 \\ \text{“}\ 13,\ & \underline{1.113\,943\,4} \\ & 2.155\,336\,1 \end{aligned}$$

Comparing this with the computation to five places, we see the source of the error.

If the numbers with which we enter the tables are affected by errors, these errors will of course increase the possible errors of the logarithms.

In determining to what degree of accuracy to carry our results, we have the following practical rule:

It is never worth while to carry our decimals beyond the limit of precision given by the tables, which limit may be a considerable fraction of the unit in the last figure of the tables.

Let us have a logarithm to five places of decimals, 1.929 49, of which we require the corresponding number. Entering the table, we perceive that the corresponding number is between 85.01 and 85.02. If this logarithm is the result of adding a number of logarithms, each of which may be in error in the way pointed out, we may suppose it probably affected by an error of half a unit in the last figure and possibly by an error of a whole unit or more. That is, its true value may be anywhere between 92.948 and 92.950.

The number corresponding to the former value is 85.012, and that corresponding to the latter 85.016. Since the numbers may fall anywhere between these limits, we assign to it a mean value of 85.014, which value, however, may be in error by two units in the last place. It is not, therefore, worth while to carry the interpolation further and to write more than five digits.

Next suppose the logarithm to be 2.021 70. Entering the table, we find in the same way that the number probably lies between the limits 105.121 and 105.126. There is therefore an uncertainty of five units in the sixth place, or half a unit in the fifth place. If the greatest precision is desired, we should write 105.124. But our last figure being doubtful by two or three units, the question might arise whether it were worth while to write it at all. As a general rule, if the sixth figure is required to be exact, we must use a six- or seven-place table of logarithms.

Still, near the beginning of the table, the probable error will be diminished by writing the sixth figure.

Now knowing that at the beginning of the table a difference of one unit in the number makes a change ten times as great in the logarithm as at the end of the table, we reach the conclusions:

In taking out a number in the first part of the table, it can never be worth while to write more than six significant figures, and very little is added to the precision by writing more than five.

In the latter part of the table it is never worth while to write more than five significant figures.

Sometimes no greater accuracy is required than can be gained by using four-figure logarithms. There is then no need of writing the last figure. If, however the printed logarithm is used without change, the fourth figure must be increased by unity whenever the fifth figure exceeds 5. When the fifth figure is exactly 5, the increase should or should not be made according as the 5 is too small or too great. To show how the case should be decided, a stroke is printed above the 5 when it is too great. In these cases the fourth figure should be used as it stands, but, when there is no stroke, it should be increased by unity.

12. Applications of Logarithms to the Computation of Annuities and Accumulations of Funds at Compound Interest.

One of the most useful applications of logarithms is to fiscal calculations, in which the value of moneys accumulating for long periods at compound interest is required.

Compound interest is gained by any fund on which the interest is collected at stated intervals and put out at interest.

As an example, suppose that $10 000 is put out at 6 per cent interest, and the interest collected semi-annually and put out at the same rate. The principal will then grow as follows:

Principal at starting	$10 000.00
Six months' interest = 3 per cent	300.00
Amount at end of 6 months	$10 300.00
Interest on this amount = 3 per cent	309.00
Amount at end of 1 year	$10 609.00
Interest on this amount = 3 per cent	318.27
Amount at end of 1½ years	$10 927.27
Interest on this amount for 6 months	327.82
Amount at end of 2 years	$11 255.09

Although in business practice interest is commonly payable semi-annually, it is in calculations of this kind commonly supposed to be collected and re-invested only at the end of each year. This makes the computation more simple, and gives results nearer to those obtained in practice, because a company cannot generally invest its income immediately. If it had to wait three months to invest each semi-annual instalment of interest collected, the general result would be about the same as if it collected interest only once a year and invested it immediately.

If r be the rate per cent per annum, the annual rate of increase will be $\frac{r}{100}$. Let us put

ρ, the annual rate of increase $= \frac{r}{100}$;

p, the amount at interest at the beginning of the time, or the principal;

a, the amount at the end of one or more years.

Then, at the beginning of first year, principal....... p
Interest during the year........................ ρp

Amount at end of year........................ $p(1+\rho)$
Interest on this amount during second year...... $\rho p(1+\rho)$
Amount at end of second year, $(1+\rho)p(1+\rho) = p(1+\rho)^2$

Continuing the process, we see that at the end of n years the amount will be

$$a = p(1+\rho)^n. \qquad (1)$$

To compute by logarithms, let us take the logarithms of both members. We then have

$$\log a = \log p + n \log(1+\rho). \qquad (2)$$

Example. Find the amount of $1250 for 30 years at 6 per cent per annum.

Here $\rho = .06$
$1+\rho = 1.06$
$\log(1+\rho) = 0.025\,306$ (end of Table I.)
30

$n \log(1+\rho)$, $0.759\,18$
$\log p$, $3.096\,91$

$\log a$, $3.856\,09$
a, $7179.50 = required amount.

Exercises.

1. Find the amount of $100 for 100 years at 5 per cent compound interest.

2. A man bequeathed the sum of $500 to accumulate at 4 per cent interest for 80 years after his death. After that time the annual interest was to be applied to the support of a student in Harvard College. What would be the income from the scholarship?

3. If the sum of one cent had been put out at 3 per cent per annum at the Christian era, and accumulated until the year 1800, what would then have been the amount, and the annual interest on this amount?

It is only requisite to give three significant figures, followed by the necessary number of zeros.

4. Solve by logarithms the problem of the horseshoeing, in which a man agrees to pay 1 cent for the first nail, 2 for the second, and so on, doubling the amount for every nail for 32 nails in all.

Note. It is only necessary to compute the amount for the 32d nail, because it is easy to see that the amount paid for each nail is 1 cent more than for all the preceding ones.

5. A man lays aside \$1000 as a marriage-portion for his new-born daughter, and invests it so as to accumulate at 8 per cent compound interest. The daughter is married at the age of 25. What does the portion amount to?

6. A man of 30 pays \$2000 in full for a \$5000 policy of insurance on his life. Dying at the age of 80, his heirs receive \$7000, policy and dividends. If the money was worth 4 per cent to him, how much have the heirs gained or lost by the investment?

7. What would have been the answer to the previous question, had the man died at the age of 40, and the amount paid been \$6000?

Other applications of the formulæ. By means of the equations (1) and (2) we may obtain any one of the four quantities a, p, ρ, and n when the other three are given.

CASE I. Given the *principal*, *rate* of interest, and *time*, to find the *amount*.

This case is that just solved.

CASE II. Given the *amount*, *time*, and *rate* per cent, to find the *principal*.

Solution. Equation (1) gives

$$p = \frac{a}{(1+\rho)^n}.$$

Taking the logarithms,

$$\log p = \log a - n \log (1+\rho),$$

by which the computation may be made.

CASE III. Given the *principal*, *amount*, and *time*, to find the *rate*.

Solution. Equation (2) gives

$$\log (1+\rho) = \frac{\log a - \log p}{n} = \frac{1}{n} \log \frac{a}{p}.$$

Example. A man wants a principal of \$600 to amount to \$1000 in 10 years. At what rate of interest must he invest it?

Solution.

$$\begin{aligned} \log a &= 3.000\,00 \\ \log p &= 2.778\,15 \\ \hline \log \frac{a}{p} &= 0.221\,85 \end{aligned}$$

$$\frac{1}{10} \log \frac{a}{p} = 0.022\,185 = \log (1+\rho).$$

Hence, from last page of logarithms,

$$1+\rho = 1.052\,41;$$

and $\qquad$ rate = 5.241,

or $5\frac{1}{4}$ per cent, nearly.

EXERCISES.

1. At what rate of interest will money double itself every ten years? Ans. 7.177.

2. At what rate will it treble itself every 15 years? Ans. 7.599.

3. A man having invested $1000, with all the interest it yielded him, for 25 years, finds that it amounts to $3386. What was the rate of interest? Ans. 5 per cent.

4. A life company issued to a man of 20 a paid-up policy for $10,000, the single premium charged being $3150. If he dies at the age of 60, at what rate must the company invest its money to make itself good? Ans. 2.93 per cent.

5. A man who can gain 4 per cent interest wants to invest such a sum that it shall amount to $5000 when his daughter, now 5 years old, attains the age of 20. How much must he invest? Ans. $2776.62.

6. How much must a man leave in order that it may amount to $1,000,000 in 500 years at 2½ per cent interest? Ans. $4.36½

7. How much if the time is 1000 years, the rate being still 2½ per cent, and the amount $1,000,000? Ans. 0.0019 of a cent.

8. A man finds that his investment has increased fivefold in 25 years. What is the average rate of interest he has gained? Ans. 6.65.

9. An endowment of $7500 is payable to a man when he attains the age of 65. What is its value when he is 45, supposing the rate of interest to be 4 per cent? Ans. $3423.

13. Accumulation of an Annuity.

It is often necessary to ascertain the present or future value of a series of equal annual payments. Thus it is very common to pay a constant annual premium for a policy of life insurance. The value of such a series of payments at any epoch is found by reducing the value of each one to the epoch, allowing for interest, and taking the sum. Supposing the epoch to be the present time, the problem may be stated as follows:

A man agrees to pay p dollars a year for n years, the first payment being due in one year, and the total number of payments n. What is the present value of all n payments?

Putting, as before, $\rho = \frac{\text{rate of interest}}{100}$, the present value of p dollars payable after y years will, by § 12, Case II., be

$$\frac{p}{(1+\rho)^y}.$$

Putting in succession, $y = 1$, $y = 2, \ldots y = n$, the sum of the present values is

$$\frac{p}{1+\rho} + \frac{p}{(1+\rho)^2} + \frac{p}{(1+\rho)^3} + \ldots + \frac{p}{(1+\rho)^n}.$$

This is a geometrical progression in which

$$\text{First term} = \frac{p}{1+\rho};$$

$$\text{Common ratio} = \frac{1}{1+\rho};$$

$$\text{Number of terms} = n.$$

By *College Algebra*, § 212, the sum of this progression will be

$$\Sigma_1 = \frac{p}{1+\rho} \cdot \frac{1-\left(\frac{1}{1+\rho}\right)^n}{1-\frac{1}{1+\rho}} = p\,\frac{(1+\rho)\quad -1}{(1+\rho)^{n+1}-(1+\rho)^n}$$

$$= \frac{p}{(1+\rho)^n} \cdot \frac{(1+\rho)^n - 1}{\rho}. \qquad (1)$$

If the first payment is to be made immediately, instead of at the end of a year, the last or nth payment will be due in $n-1$ years, and the progression will be

$$p + \frac{p}{1+\rho} + \frac{p}{(1+\rho)^2} + \ldots + \frac{p}{(1+\rho)^{n-1}}.$$

We find the sum of the geometric progression to be

$$\Sigma_2 = p\,\frac{(1+\rho)^n - 1}{(1+\rho)^n - (1+\rho)^{n-1}}. \qquad (2)$$

EXERCISES.

1. What is the present value of 15 annual payments of \$85 each, of which the first is due in one year, the rate being 5 per cent?

We find by substitution

$$\text{Present value} = 85\,\frac{1.05^{15}-1}{1.05^{16}-1.05^{15}} = \frac{85}{1.05^{15}} \cdot \frac{1.05^{15}-1}{.05}$$

$$= \frac{1700\,(1.05^{15}-1)}{(1.05)^{15}}.$$

log 1.05, 0.021 189
15
log 1.05^{15}, 0.317 84

1.05^{15}, 2.078 95
$1.05^{15} - 1$, 1.078 95
log, 0.033 00
co-log 1.05^{15}, 9.682 16
log 1700, 3.230 45
log value, 2.945 61

Value, \$882.28

2. The same thing being supposed, what would be the present value if the rate of interest were 4 per cent? Ans. \$945.80.

3. What is the present value of 25 annual payments of \$1000 Ans. \$17,935.
each, the first due immediately, if the rate of interest is 3 per cent?

4. A debtor owing \$10,000 wishes to pay it in 10 equal annual instalments, the first being payable immediately. If the rate of interest is 6 per cent, how much should each payment be?

Ans. \$1281.76.

NOTE. This problem is the reverse of the given one, since, in the equation (2), we have given $\Sigma_2 = 10\,000$, $\rho = 0.06$, and $n = 10$, to find p.

5. The same thing being supposed, what should be the annual payment in case the payments should begin in a year?

Ans. \$1358.69.

Perpetual annuities. If the rate of interest were zero, the present value of an infinity of future payments would be infinite. But with any rate of interest, however small, they are finite. For if, in the first equation (1), we suppose n infinite, $\left(\frac{1}{1+\rho}\right)^n$ will converge toward zero, and we shall have

$$\Sigma = \frac{p}{(1+\rho)\left(1 - \frac{1}{1+\rho}\right)} = \frac{p}{\rho}. \qquad (3)$$

This result admits of being put into a concise form, thus:

Since Σ is the present value of the perpetual annuity p, the annual interest on this value will be $\rho\Sigma$. But the equation (3) gives

$$\rho\Sigma = p.$$

Hence:

The present value of a perpetual annuity is the sum of which the annuity is the annual interest.

Example. If the rate of interest were $3\frac{1}{2}$ per cent, the present value of a perpetual annuity of \$70 would be \$2000.

EXERCISES.

1. A government owing a perpetual annuity of \$1000 wishes to pay it off by 10 equal annual payments. If the rate of interest is 4 per cent, what should be the amount of each payment?

Ans. \$3082.30.

2. A government bond of \$100 is due in 15 years with interest at 6 per cent. The market rate of interest having meanwhile fallen to $3\frac{1}{2}$ per cent, what should be the value of the bond?

NOTE. We find, separately, the present value of the 15 annual instalments of interest, and of the principal.

TABLE II.

MATHEMATICAL CONSTANTS.

14. In this table is given a collection of constant quantities which frequently occur in computation, with their logarithms.

The logarithms are given to more than five decimals, in order to be useful when greater accuracy is required. When used in five-place computations, the figures following the fifth decimal are to be dropped, and the fifth decimal is to be increased by unity in case the figure next following is 5 or any greater one.

TABLES III. AND IV.

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

15. By means of these tables the logarithms of the six trigonometric functions of any angle may be found.

The logarithm of the function instead of the function itself is given, because the latter is nearly always used as a factor.

We begin by explaining Table IV., because Table III. is used only in some special cases where Table IV. is not convenient.

I. *Angles less than* 45°. If the angle of which a function is sought is less than 45°, we seek the number of degrees at the top of the table and the minutes in the left-hand column. Then in the line opposite these minutes we find successively the sine, the tangent, the cotangent, and the cosine of the angle, as given at the heading of the page.

Example.

$$\log \sin 31^\circ\, 27' = 9.717\,47;$$
$$\log \tan 31^\circ\, 27' = 9.786\,47;$$
$$\log \operatorname{cotan} 31^\circ\, 27' = 0.213\,53;$$
$$\cos 31^\circ\, 27' = 9.931\,00.$$

The sine, tangent, and cosine of this angle being all less than unity, the true mantissæ of the logarithm are negative; they are therefore increased by 10, on the system already explained.

If the secant or cosecant of an angle is required, it can be found by taking the arithmetical complement of the cosine or sine. It is shown in trigonometry that

$$\text{secant} = \frac{1}{\text{cosine}},$$

and

$$\text{cosecant} = \frac{1}{\text{sine}}.$$

Therefore $\log \text{secant} = 0 - \log \text{cosine} = \text{co-log cosine};$

$$\log \text{cosec} = 0 - \log \text{sine} = \text{co-log sine}.$$

We thus find

$$\log \sec 31^\circ\, 27' = 0.069\,00;$$
$$\log \operatorname{cosec} 31^\circ\, 27' = 0.282\,53.$$

After each column, upon intermediate lines, is given the difference between every two consecutive logarithms, in order to facilitate interpolation.

In the case of tangent and cotangent, only one column of differences is necessary for both functions.

If we use no fractional parts of minutes, no interpolation is necessary; but if decimals of a minute are employed, we can interpolate precisely as in taking out the logarithms of numbers.

Where the differences are very small they are sometimes omitted.

Tables of proportional parts are given in the margin, the use of which is similar to those given with the logarithms of numbers.

Example 1. To find the log sin of 31° 27′.7.

$$\begin{array}{lr} \text{We have from the tables, } \log\sin 31^\circ\ 27' & = 9.717\ 47 \\ \text{Under diff. 20, P.P. for 7,} & 14 \\ \hline \log\sin 31^\circ\ 27'.7 & = 9.717\ 61 \end{array}$$

Ex. 2. To find log cot 15° 44′.34.

$$\begin{array}{lr} \text{The tables give } \log\cot 15^\circ\ 44' & = 0.550\ 19 \\ \text{Under diff. 48, opposite 0.3, P.P.,} & -\ 14.4 \\ \text{“ \qquad “ \qquad } 0.4 \div 10, & -\ 1.9 \\ \hline \log\cot 15^\circ\ 44'.34, & 0.550\ 03 \end{array}$$

Since the tabular quantity diminishes as the angle increases, the proportional parts are subtractive.

EXERCISES.

Find from the tables:

1. log cot 43° 29′.3;
2. log tan 43° 29′.3;
3. log cos 27° 10′.6;
4. log sin 27° 10′.6;
5. log tan 12° 9′.43;
6. log cot 12° 9′.43.

In the case of sines and tangents of small angles the differences vary so rapidly that in most cases the exact difference will not be found in the table of proportional parts. In this case, if the proportional parts are made use of, a double interpolation will generally be necessary to find the fraction of a minute corresponding to a given sine or tangent. If only tenths of minutes are used, an expert computer will find it as easy to multiply or divide mentally as to refer to the table.

II. *Angles between* 45° *and* 90°. It is shown in trigonometry that if we compute the values of the trigonometric functions for the first 45°, we have those for the whole circle by properly exchanging them in the different parts of the circle. First, if we have

$$\alpha + \beta = 90,$$

then α and β are complementary functions, and

$$\sin \beta = \cos \alpha;$$
$$\tan \beta = \text{cotan } \alpha.$$

Therefore if our angle is between 45° and 90°, we may find its complement. Entering the table with this complement, the complementary function will then be the required function of the angle.

Example. To find the sine of 67° 23′, we may enter the table with 22° 37′ (= 90° − 67° 23′) and take out the *cosine* of 22° 37′, which is the required *sine* of 67° 23.

To save the trouble of doing this, the complementary angles and the complementary denominations of the functions are given at the bottom of the page.

The minutes corresponding to the degrees at the bottom are given on the right hand. Therefore:

To find the trigonometric functions corresponding to an angle between 45° and 90°, we take the degrees at the bottom of the page and the minutes in the right-hand column. The values of the four functions log sine, log tangent, log cotangent, and log cosine, as read at the bottom of the page, are then found in the same line as the minutes.

Example 1. For 52° 59′ we find

$$\log \sin = 9.902\,25;$$
$$\log \tan = 0.122\,62;$$
$$\log \cot = 9.877\,38;$$
$$\log \cos = 9.779\,63.$$

Ex. 2. To find the trigonometric functions of 77° 17′.28.

	sin.	tan.	cot.	cos.
77° 17′	9.989 21	0.646 53	9.353 47	9.342 68
P.P. for 0.2	+ 0.6	+ 11.8	− 11.8	− 11.2
“ 0.08	+ 0.2	+ 4.7	− 4.7	− 4.5
	9.989 22	0.646 70	9.353 30	9.342 52

Then log sec = co-log cos = 0.657 48;
log cosec = co-log sin = 0.010 78.

Exercises.

Find the logarithms of the six functions of the following angles:

1. 45° 50′.74;
2. 48° 49′.37;
3. 74° 0′.68;
4. 83° 59′.62.

III. *When the angle exceeds* 90°.

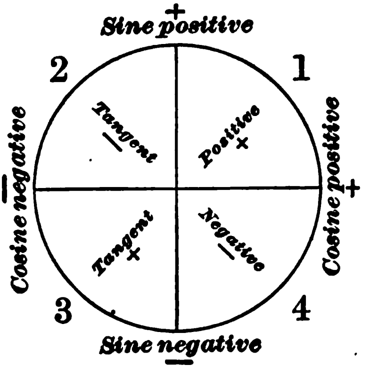
Rule. *Subtract from the angle the greatest multiple of* 90° *which it contains.*

If this multiple is 180°, *enter the table with the excess of the angle over* 180° *and take out the functions required, as if this excess were itself the angle.*

If the multiple is 90° *or* 270°, *take out the complementary function to that required.*

By then assigning the proper algebraic sign, as shown in trigonometry, the complete values of the function will be obtained.

The computer should be able to assign the proper algebraic sign according to the quadrant, without burdening his memory with the special rules necessary in each case. This he can do by carrying in his mind's eye the following scheme. He should have at command the arrangement of the four quadrants as usually represented in trigonometry, so as to know, when an angle is stated, where it will fall relatively to the horizontal and vertical lines through the centre of the circle. Then, in the case of—



Sine or *cosecant.* If the angle is *above* the horizontal line (which it is between 0° and 180°), the sine is *positive;* if *below*, *negative.*

Cosine or *secant.* If the angle is to the *right* of the vertical central line (as it is in the first and fourth quadrants), the cosine and secant are *positive;* if to the *left* (as in the second and third quadrants), *negative.*

Tangent or *cotangent.* Through the opposite first and third quadrants, *positive;* through the opposite second and fourth quadrants, *negative.*

Example 1. To find the tangent and cosine of 122° 44′. Subtracting 90°, we enter the table with 32° 44′ and find

$$\log \cot\ 32° 44' = 0.191\,92;$$
$$\log \sin\ 32° 44' = 9.732\,98.$$

Therefore, writing the algebraic sign before the logarithm, we have

$$\log \tan 122° 44' = -\ 0.191\,92;$$
$$\log \cos 122° 44' = -\ 9.732\,98.$$

Ex. 2. To find the sine and cotangent of 322° 58′.

Entering the table with 52° 58′ = 322° 58′ − 270°, and taking out the complementary functions, we find

$$\log \sin 322° \, 58' = -\ 9.779\,80;$$
$$\log \cot 322° \, 58' = -\ 0.122\,36.$$

Ex. 3. To find the sine and tangent of 253° 5′.

Entering with 73° 5′, we take out the sine and tangent, finding

$$\log \sin 253° \, 5' = -\ 9.890\,79;$$
$$\log \tan 253° \, 5' = +\ 0.516\,93.$$

Ex. 4. To find the six trigonometric functions of 152° 38′. We have

$$\begin{aligned} \log \sin 152° \, 38' &= \log \cos 62° \, 38' \ \textit{pos.} &&= +\ 9.662\,46; \\ \log \cos 152° \, 38' &= \log \sin 62° \, 38' \ \textit{neg.} &&= -\ 9.948\,45; \\ \log \tan 152° \, 38' &= \log \cot 62° \, 38' \ \textit{neg.} &&= -\ 9.714\,01; \\ \log \cot 152° \, 38' &= \log \tan 62° \, 38' \ \textit{neg.} &&= -\ 0.285\,99; \\ \log \sec &= \text{co-log} \cos &&= -\ 0.051\,55; \\ \log \text{cosec} &= \text{co-log} \sin &&= +\ 0.337\,54. \end{aligned}$$

EXERCISES.

Find the six trigonometric functions of the following angles:

276° 29′.3;
66° 0′.5;
96° 59′.8;
252° 20′.3;
318° 10′.7;
− 25° 22′.2;
−155° 30′.7.

16. Method of Writing the Algebraic Signs.

As logarithms are used in computation, they may always be considered positive. It is true that the logarithms of numbers less than unity are in reality negative, but, for convenience in calculation, we increase them by 10, so as to make them positive.

The number corresponding to a given logarithm may, in computation, be positive or negative. There are two ways of distinguishing the algebraic sign of the number, between which the computer may choose for himself.

I. *Write the algebraic sign of the number before the logarithm.* As usually interpreted, the algebraic sign written thus would apply to the logarithm, which it does not. It is therefore necessary for the computer to bear in mind that the sign belongs, not to the logarithm, as written, but to the number.

II. *Write the letter n after the logarithm when the number is negative.* This plan is theoretically the best, but, should the computer accidentally omit the letter, the number will be treated as positive, and a mistake will be made. It therefore requires vigilance on his part. An improvement would be to write a letter not likely to be mistaken for *n*, *s* for instance, after all positive logarithms.

17. To Find the Angle Corresponding to a Given Trigonometric Function.

Disregarding algebraic signs, there will always be four angles corresponding to each function, one in each quadrant. These angles will be:

The smallest angle, as found in the table;
This angle increased by 180°;
The complementary angle increased by 90°;
The complementary angle increased by 270°.

For instance, for the angle of which log tan is 0.611 92, we find 76° 16′. But we should get this same tangent for 103° 44′, 256° 16′, and 283° 44′.

Of the four functions corresponding to the four angles, two will always be positive and two negative; so that, in reality, there will only be two angles corresponding to a function of which both the sign and the absolute value are given. These values are found by selecting from the four possible ones the two for which the functions have the given algebraic sign. After selecting them, they may be checked by the following theorems, which are easily deduced from the relations between the values of each function as given in trigonometry:

The sum of the two angles corresponding to the same sine is 180° *or* 540°.

The sum of the two angles corresponding to the same cosine is 360°.

The difference of the two angles corresponding to the same tangent is 180°.

Which of the two possible angles is to be chosen depends upon the conditions of the problem or the nature of the figure to which the angle belongs. If neither the conditions nor the figure decide the question, the problem is essentially ambiguous, and either or both angles are to be taken.

EXERCISES.

Find the pairs of values of the angle α from the following values of the trigonometric functions:

1. $\log \sin \alpha = +\,9.902\,43$;
2. $\log \sin \alpha = -\,9.902\,43$;
3. $\log \cos \alpha = +\,9.902\,43$;
4. $\log \cos \alpha = -\,9.902\,43$;
5. $\log \tan \alpha = +\,0.143\,16$;
6. $\log \tan \alpha = -\,0.143\,16$;
7. $\log \cot \alpha = +\,0.143\,16$;
8. $\log \cot \alpha = -\,0.143\,16$;
9. $\log \tan \alpha = -\,9.024\,81$;
10. $\log \tan \alpha = -\,0.975\,19$;
11. $\log \tan \alpha = +\,0.975\,19$;
12. $\log \sec \alpha = +\,0.221\,06$;
13. $\log \sec \alpha = -\,0.221\,06$;
14. $\log \sec \alpha = -\,0.099\,20$;
15. $\log \sec \alpha = +\,0.123\,46$;
16. $\log \sin \alpha = +\,8.990\,30$;
17. $\log \sin \alpha = -\,8.990\,30$;
18. $\log \cos \alpha = +\,9.218\,67$;
19. $\log \cos \alpha = -\,9.218\,67$;
20. $\log \tan \alpha = -\,9.136\,90$;
21. $\log \tan \alpha = +\,9.136\,90$;
22. $\log \cot \alpha = +\,9.136\,90$.

18. Cases when the Function is very Small or Great.

When the angle of which we are to find the functions approaches to zero, the logarithms of the sine, tangent, and cotangent vary so rapidly that their values to five figures cannot be readily interpolated. The same remark applies to the cosine, cotangent, and tangent of angles near 90° or 270°. The mode of proceeding in these cases will depend upon circumstances.

In the use of five-place logarithms, there is little advantage in carrying the computations beyond tenths of minutes, though the hundredths may be found when the tangent or cotangent is given. Where greater accuracy than this is required, six- or seven-place tables must be used.

If the angles are only carried to tenths of minutes, there is no necessity for taking out the sine, tangent, or cotangent to more than four decimals when the angle is less than 3°, and three decimals suffice for angles less than 30′. The reason is that this number of decimals then suffice to distinguish each tenth of minute.

When the decimals are thus curtailed, an expert computer will be able to perform the multiplication and division for the tenths of minutes mentally. If, however, this is inconvenient, the following rule may be applied.

To find the log sine or log tangent of an angle less than 2° to four places of decimals:

RULE. *Enter the table of logarithms of numbers with the value* *of the angle expressed in minutes and tenths, and take out the logarithm.*

To this logarithm add the quantity 6.4637.

The sum will be the log sine, and the log tangent may be assumed to have the same value.

Example 1. To find log sin 1° 22′.6.

1° 22′.6 = 82′.6

log 82′.6 = 1.9170

constant, 6.4637

log sin 1° 22′.6, 8.3807

This rule is founded on the theorem that the sines and tangents of very small arcs may be regarded as equal to the arcs themselves. Since, in using the trigonometric functions, the radius of the circle is taken as unity, an arc must be expressed in terms of the unit radius when it is to be used in place of its sine or tangent. Now, it is shown in trigonometry that the unit radius is equal to 57°.2958 or 3437′.747 or 206 264″.8. Hence we must divide the number of angular units in the angle by the corresponding one of these coefficients to obtain the length of the corresponding arcs in unit radius. Now,

log 3437.747 = 3.5363

co-log....... 6.4637

which may be added instead of subtracting the logarithm.

To find the cosine of an angle very near 90°, we find the sine of its complement, which will then be a very small angle, positive or negative.

EXERCISES.

Find to four places of decimals:

1. log sin 22′.73;
2. log sin 1° 1′.12;
3. log cos 90° 0′.78;
4. log tan 88° 59′.35;
5. log cot 90° 28′.76;
6. log cos 89° 22′.23;
7. log sin 0° 0′.25.

If an angle corresponding to a given sine or tangent is required, the rule is:

From the given log sine or tangent subtract 6.4637 *or add* 3.5363. *The result is the logarithm of the number of minutes.*

Of course this rule applies only to angles less than 2°, in the value of which only tenths of minutes are required.

EXERCISES.

Find α from:

1. $\log \sin \alpha = 7.2243$;
2. $\log \cot \alpha = 2.8816$;
3. $\log \tan \alpha = -2.8816$;
4. $\log \cos \alpha = 6.9218$.

When the small angle is given in seconds. Although the computer may take out his angles to tenths of minutes, cases often arise in which a small angle is given in seconds, or degrees, minutes, and seconds, and in which the trigonometric function is required to five decimals. In this case the preceding method may not always give accurate results, because the arc and its sine or tangent may differ by a greater amount than the error we can admit in the computation.

Table II. is framed to meet this case. The following are the quantities given:

In the second column: The argument, in degrees and minutes, as already explained for Table III.

In the first column: This argument reduced to seconds. From this column the number of seconds in an arc of less than 2°, given in degrees, minutes, and seconds, may be found at sight.

Example. How many seconds in $1^\circ\,28'\,39''$? In the table, before $1^\circ\,28'$, we find $5280''$, which being increased by $39''$ gives $5319''$, the number required.

Col. 3. The logarithm of the sine of the angle. This is the same as in Table III.

Col. 4. The value of log sine *minus* log arc; that is, the difference between the logarithm of the sine and the logarithm of the number of seconds in the angle.

Col. 5. The same quantity for the tangent.

Cols. 6 and 7. The complements of the preceding logarithms, distinguished by accents.

The use of the tables is as follows.

To find the sine or tangent of an angle less than 2°:

Express the angle in seconds by the first two columns of the table.

Write down the logarithm in column S or column T, according as the sine or a tangent is required.

Find from Table I. the logarithm of the number of seconds.

Adding this logarithm to S or T, the sum will be the log sine or log tangent.

Example. Find $\log \sin 1^\circ\,2'\,47''.9$.

$$\begin{array}{rr} & S,\ 4.685\,55 \\ 1^\circ\,2'\,47''.9 = 3767''.9;\ \log, & \underline{3.576\,10} \\ \log \sin 1^\circ\,2'\,47''.9, & 8.261\,65 \end{array}$$

To find the arc corresponding to a given sine or tangent:

Find in the column L. sin. the quantity next greater or next smaller than the given logarithm.

Take the corresponding value of S′ or T′ according as the given function is a sine or tangent, and add it to the given function.

The sum is the logarithm of the number of seconds in the required angle.

Example. Given $\log\tan x = 8.401\,25$, to find x.

$$\begin{array}{rr} \log\tan x, & 8.401\,25 \\ \mathrm{T}', & \underline{5.314\,33} \\ \log x, & 3.715\,58 \end{array}$$

$x = 5194''.9 = 1^\circ\ 26'\ 34''.9$, from col. 2.

EXERCISES.

Find:

1. $\log\sin 0^\circ\ 20'\ 20''.25$;
2. $\log\tan 0^\circ\ \ 0'\ \ 1''.2273$;
3. $\log\sin 1^\circ\ 59'\ 22''.7$;
4. $\log\tan 1^\circ\ \ 0'\ 59''.7$.

Find x from:

1. $\log\tan x = 8.427\,96$;
2. $\log\tan x = 7.427\,96$;
3. $\log\tan x = 6.427\,96$;
4. $\log\sin x = 5.354\,35$;
5. $\log\sin x = 4.226\,19$;
6. $\log\sin x = 8.540\,78$.

When the cosine or cotangent of an angle near 90° or 270° is required, we take its difference from 90° or 270°, and find the complementary function by the above rules.

Remark. The use of the logarithms of the trigonometric functions is so much more extensive than that of the functions themselves that the prefix "log" is generally omitted before the designation of the logarithmic function, where no ambiguity will result from the omission.

TABLE V.

NATURAL SINES AND COSINES.

19. This table gives the actual numerical values of the sine and cosine for each minute of the quadrant.

To find the sine or cosine corresponding to a given angle less than 45°, we find the degrees at the top of a pair of columns and the minutes on the left.

In the two columns under the degrees and in the line of minutes we find first the sine and then the cosine, as shown at the head of the column.

A decimal point precedes the first printed figure in all cases, except where the printed value of the function is unity.

If the given angle is between 45° and 90°, find the degrees at the bottom and the minutes at the right.

Of the two numbers above the degrees, the right-hand one is the sine and the left-hand one the cosine.

For angles greater than 90° the functions are to be found according to the precepts given in the case of the logarithms of the sines and tangents.

TABLE VI.

ADDITION AND SUBTRACTION LOGARITHMS.

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20. Addition and subtraction logarithms are used to solve the problem: *Having given the logarithms of two numbers, to find the logarithm of the sum or difference of the numbers.*

The problem can of course be solved by finding the numbers corresponding to the logarithms, adding or subtracting them, and taking out the logarithm of their sum or difference. The table under consideration enables the result to be obtained by an abbreviated process.

I. *Use in addition.* The principle on which the table is constructed may be seen by the following reasonings. Let us put

$$S = a + b,$$

a and b being two numbers of which the logarithms are given. We shall have

$$S = a\left(1 + \frac{b}{a}\right) = a\,(1 + x);$$

putting, for clearness, $x = \frac{b}{a}$.

We then have

$$\log S = \log a + \log\,(1 + x).$$

Since $\log a$ and $\log b$ are both given, we can find $\log x$ from the equation

$$\log x = \log b - \log a,$$

which is therefore a known quantity.

Now, for every value of $\log x$ there will be one definite value of each of the quantities x, $1 + x$, and $\log\,(1 + x)$. Therefore a table may be constructed showing, for every value of $\log x$, the corresponding value of $\log\,(1 + x)$.

Such a table is Table VI.

The argument, in column A, being $\log x$, the quantity B in the table is $\log\,(1 + x)$.

Example. $\log 0.25 = 9.397\,94$.

Entering the table with $A = 9.397\,94$, we find

$$B = 0.096\,91,$$

which is the logarithm of 1.25.

Therefore, entering the table with log x as the argument, we take out log $(1 + x)$, which added to log a will give log S.

We have therefore the following precept for using the table in addition:

Take the difference of the two given logarithms.

Enter the table with this difference as the argument A, and take out the quantity B.

Adding B to the subtracted logarithm, the sum will be the required logarithm of the sum.

It is indifferent which logarithm is subtracted, but convenience in interpolating will be gained by subtracting the greater logarithm from the lesser increased by 10. The number B will then be added to the greater logarithm.

Example. Given log $m = 1.629\,74$, log $n = 2.203\,86$; find log $(m + n)$.

The required logarithm is found in either of the following two ways:

log m,	1.629 74	(1)	log B,	0.676 76	(4)
log n,	2.203 86	(2)	log m,	1.629 74	(1)
B,	0.102 64	(4)	log n,	2.203 86	(2)
$A = \log m \div n$,	9.425 88	(3)	$\log n \div m$,	0.574 12	(3)
log $(m + n)$,	2.306 50	(5)	log $(m + n)$,	2.306 50	(5)

The figures in parentheses show the order in which the numbers are written.

EXERCISES.

Log a and log b having the following values, find log $(a + b)$.

1. log $a = 1.700\,37$; log $b = 1.921\,69$.
2. log $a = 0.624\,60$; log $b = 1.881\,26$.
3. log $a = 9.791\,86$; log $b = 0.322\,09$.
4. log $a = 9.601\,62$; log $b = 1.306\,06$.
5. log $a = 0.792\,90$; log $b = 9.221\,27$.
6. log $a = 0.601\,32$; log $b = 9.001\,68$.
7. log $a = 2.796\,43$; log $b = 3.981\,86$.

II. *Use in subtraction.* The problem is, having given log a and log b, to find the logarithm of

$$D = a - b.$$

We have

$$D = b\left(\frac{a}{b} - 1\right);$$

whence

$$\log D = \log b + \log\left(\frac{a}{b} - 1\right).$$

Since $\log \frac{a}{b}$ is found by subtracting $\log b$ from $\log a$, if we can find $\log \left(\frac{a}{b} - 1\right)$ from $\log \frac{a}{b}$, the problem will be solved.

From the construction of the table already explained, if we have

$$B = \log \frac{a}{b},$$

we must have

$$A = \log \left(\frac{a}{b} - 1\right).$$

We now have the following precept for subtraction:

Subtract the lesser of the given logarithms from the greater.

Enter the table so as to find the difference of the logarithms in the numbers B of the table.

Add the corresponding value of A to the lesser of the given logarithms. The sum will be the logarithm of the difference.

Example. Find $\log (n - m)$ in the example of the preceding section.

$$\begin{array}{rrl}
\log n, & 2.203\,86 & (1) \\
\log m, & 1.629\,74 & (2) \\
A, & \underline{0.439\,45} & (4) \\
\log \frac{n}{m} = B, & \underline{0.574\,12} & (3) \\
\log (n - m), & 2.069\,19 & (5)
\end{array}$$

Exercises.

Find the logarithms of the differences of the quantities a and b in the preceding section.

Remark. In the use of addition and subtraction logarithms, the precepts apply to *numerical* sums and differences, without respect to the algebraic signs of the quantities. For example, the algebraic difference between $+ 1473$ and $- 29\,462$ is to be found by addition, and the algebraic sum of a positive and negative quantity by subtraction.

Case where the quotient is large. Near the end of the table, A and B become nearly equal; the structure of the table is therefore changed so as to simplify its use. It is evident that if b is very small compared with a, the logarithms of $a + b$ and $a - b$ will not differ much from the logarithm of a itself. Hence, in this case, we shall have smaller numbers to use if we can find the quantity which must be added to $\log a$ to give $\log (a + b)$, or subtracted from log a to give log $(a - b)$. Now, the equations already written give, when $a > b$,

$$\log a = \log b + A,$$
$$\log (a + b) = \log b + B;$$

whence, by subtraction,

$$\log (a + b) - \log a = B - A,$$

or $$\log (a + b) = \log a + B - A. \quad \text{(with Arg. } A\text{)}$$

We find in the same way,

$$\log (a - b) = \log a - (B - A). \quad \text{(with Arg. } B\text{)}$$

Now, whenever $\log a - \log b$ is greater than 1.65, we shall find it more convenient to take out $B - A$ from the table than either A or B. We notice that the last two figures of B in this part of the table vary slowly, and we need only attend to them in interpolating. For instance, in the horizontal line corresponding to $A = 1.66$ we find:

for $A =$	$B =$	$B - A =$
1.660 00;	1.669 40;	.009 40;
.661 00;	.670 38;	.009 38;
.662 00;	.671 36;	.009 36;
.663 00;	.672 33;	.009 33;
.664 00;	.673 31;	.009 31;
.665 00;	.674 29;	.009 29;
etc.	etc.	etc.

The interpolation of $B - A$ is now very easy whether the quantity given is A or B. We note that $B - A$ has but three significant figures, of which the first is found in column zero, and the other two are the last two figures of B as printed.

As an example, let us find $\log (a + b)$ from

$$\log a = 2.791\,63$$
$$\log b = 1.128\,19$$
$$A = 1.663\,44$$

Entering the table with this value of A, we find by column 0 that $B - A$ falls between .009 40 and .009 19. Following the horizontal line $A = 1.66$ to column 3 and interpolating the last two figures between 33 and 31 for .44, with the difference $-$ 2, we find

$$B - A = .009\,32$$

Then $$\log a = 2.791\,63$$

$$\log (a + b) = 2.800\,95$$

Next, if $\log (a - b)$ is required, we have to find the difference 1.663 44 in the part B of the table. We find in the table:

for $B = 1.662\,55$; $B - A = .009\,55$;
for $B = 1.663\,53$; $B - A = .009\,53$.

Therefore

for $B = 1.663\,44$; $B - A = .009\,53$.

Subtracting this from log a, we have

$$\log (a - b) = 2.782\,10.$$

EXERCISES.

Find $\log (a + b)$ and $\log (a - b)$ from:

8. $\log a = 0.367\,02$; $\log b = 8.462\,83$.
9. $\log a = 0.001\,26$; $\log b = 8.329\,07$.
10. $\log a = 2.069\,23$; $\log b = 0.110\,85$.
11. $\log a = 5.807\,35$; $\log b = 3.838\,09$.

For values of A and B greater than 2.00, the table is so arranged that no interpolation at all is necessary. The computer has only to find what value of A or B given in the table comes *nearest* his value of $\log a - \log b$ and take the corresponding value of $B - A$. He must remember that column A is to be entered for addition, and B for subtraction.

In this part of the table A and B are given to fewer than five decimals; because five decimals are not necessary to give $B - A$ with accuracy. The nearer the end of the table is approached, the fewer the decimals necessary in taking the difference.

Example. Find $\log (a + b)$ and $\log (a - b)$ from

$$\begin{aligned} \log a &= 1.265\,32 \\ \log b &= \underline{9.222\,30} \\ \log a - \log b,\ & 2.043\,02 \end{aligned}$$

Entering column A with this difference, we find the nearest tabular value of A to be 2.0425, to which corresponds $B - A = .003\,92$. Hence

$$\log (a + b) = 1.265\,32 + .003\,92 = 1.269\,24.$$

Entering column B with the same difference, we find $B - A = .003\,95$; whence

$$\log (a - b) = 1.265\,32 - .003\,95 = 1.261\,37.$$

EXERCISES.

Find $\log (a + b)$ and $\log (a - b)$ from:

1. $\log a = 4.069\,05$; $\log b = 2.001\,32$.
2. $\log a = 3.926\,93$; $\log b = 1.201\,59$.
3. $\log a = 3.061\,64$; $\log b = 0.126\,15$.
4. $\log a = 1,22C\,68$; $\log b = 7.321\,56$.
5. $\log a = 0.693\,17$; $\log b = 6.010\,23$.
6. $\log a = 2.306\,20$; $\log b = 7.023\,01$.

Case of nearly equal numbers. Near the beginning of the table the reverse is true: it is not possible to find A with accuracy to five places of decimals. But here the value of A taken from the tables, though it be found to only two, three, or four places of decimals, will give as accurate a result as the computation of a and b to five places will admit of. Let us suppose, for example, that we have to find $\log(a-b)$ from

$$\begin{aligned} \log a &= 9.883\,15 \\ \log b &= \underline{9.882\,96} \\ B &= 0.000\,19 \end{aligned}$$

We find $A = 6.64 - 10;$

whence $\log(a-b) = 6.52 - 10.$

We note that the value of A may be 6.63 or 6.65 as well as 6.64, so that the result cannot be carried beyond two decimals. To show that these two are as accurate as the work admits of, we find the natural numbers a and b from Table I.

$$\begin{aligned} a &= 0.764\,10 \\ b &= \underline{0.763\,77} \\ a-b &= 0.000\,33 \end{aligned}$$

Since $a-b$ has but two significant figures, and the first of these is less than 5, two figures in the logarithm are all that can be accurate.

TABLE VII.

SQUARES OF NUMBERS.

21. By means of this table the square of any number less than 1000 may be found at sight, and that of any number less than 10 000 by a simple and easy interpolation.

The first page gives the squares of the first 100 numbers, which it is often convenient to have by themselves.

On the second and third pages (98 and 99) the hundreds of the number to be squared are found at the tops of the several columns, and the tens and units in the left-hand column. The first three or four figures of the square are in the column under the hundreds, and opposite the tens and units, and the last two figures on the right of the page after the column 9 ◆ ◆

Examples. The square of 634 is 401 956;
" " 329 " 108 241;
" " 265 " 70 225;
" " 153 " 23 409;
" " 999 " 998 001.

The same table may be used for any number of three significant figures by attention to the position of the decimal-point. Thus:

$$
\begin{aligned}
51100^2 &= 2\,611\,210\,000;\\
511^2 &= 261\,121;\\
51.1^2 &= 2\,611.21;\\
5.11^2 &= 26.1121;\\
0.511^2 &= 0.261\,121.
\end{aligned}
$$

When there are four significant figures, an interpolation may be executed in several ways. If n be the nearest number the square of which is found in the table, and h the excess of the given number over this, so that $n + h$ is the number whose square is required, we shall have

$$
\begin{aligned}
(n+h)^2 = n^2 + 2nh + h^2 &= n^2 + h\,(2n+h)\\
&= n^2 + h\,(N+n);
\end{aligned}
$$

where $N = n + h$, the given number.

We may therefore find the square of 257.4 in the following way:

$$\begin{array}{rl} 257^2 = & 66\,049 \\ 514.4 \times .4 = & \underline{205.76} \\ (257.4)^2 = & 66\,254.76 \end{array}$$

To find the square of 9037 we proceed thus:

$$\begin{array}{ll} 9037 & \\ \underline{9030^2} & = 81\,540\,900 \\ 18067 \times 7 & = \underline{126\,469} \\ 9037^2 & = 81\,667\,369 \end{array}$$

In many cases only one more figure will be required in the square than in the given number. The square can then be interpolated with all required accuracy by the differences, the last two figures of which are found in the last column of the table, while the remaining figures are found by taking the difference between two consecutive numbers in the principal column.

To return to the last example, we find the difference between 257^2 and 258^2 to be 515, the first figure being the difference between 660 and 665, and the last two, 15, in the last column. Then

$$\begin{array}{rl} 257^2 = & 66\,049 \\ 515 \times 0.4 = & \underline{206} \\ (257.4)^2 = & 66\,255 \end{array}$$

—which is correct to the nearest unit.

It will be remarked that the two methods are substantially the same when only five figures are sought in the result. The substantial identity rests upon the general theorem that

The difference of the squares of two consecutive numbers is equal to the sum of the numbers.

We prove this theorem thus:

$$(n+1)^2 - n^2 = 2n + 1 = n + (n+1).$$

When the tabular difference is taken in the way already described, it will often happen that the difference between the numbers in the columns of hundreds is to be diminished by unity. Thus, although $4173 - 4160 = 13$, the difference between 645^2 and 646^2 is not 1391, but 1291. These cases are noted by the asterisk after the number in the last column.

The squares of numbers of more than four figures may be found in the same way, but in such cases it will generally be easier to use logarithms than the table of squares.

TABLE VIII.

TO CONVERT HOURS, MINUTES, AND SECONDS INTO DECIMALS OF A DAY, AND *VICE VERSA*.

22. The familiar method of solving this problem is to convert the seconds into decimals of a minute, and the minutes into decimals of an hour, by dividing by 60, and then the hours into decimals of a day by dividing by 24. The reverse problem is solved by multiplying by 24, 60, and 60.

Table VIII. enables us to perform these operations without division. Column *D* gives each hundredth of a day, but its numbers may also be regarded as ten thousandths or millionths of a day, according to which of the following three columns is used. In column *H.M.S.* are found the hours, minutes, and seconds corresponding to these hundredths. In the next column is one hundredth of column *H.M.S.*, or the minutes and seconds in the number of ten thousandths of a day in column *D*. Finally, column $\frac{H.M.S.}{100^2}$ shows the number of seconds in the number of millionths of a day found in column *D*.

Example. To convert $0^d.532\,946$ into hours, minutes, and seconds.

$$\begin{array}{rcr}
0^d.53 & = & 12^h\,43^m\,12^s \\
.002\,9 & = & 4^m\,10^s.56 \\
.000\,046 & = & 3^s.97 \\
\hline
0^d.532\,946 & = & 12^h\,47^m\,26^s.53
\end{array}$$

It will be seen that we divide the figures of the given decimal of a day into pairs, and enter the three columns of time with these three pairs in succession.

If seven decimals are given, we may interpolate the last number, as in taking out a logarithm.

Example. Convert $0^d.050\,762\,7$.

$$\begin{array}{lr}
0^d.05 = & 1^h\,12^m\,0^s \\
.000\,7 = & 1^m\,0^s.48 \\
.000\,062 = & 5^s.36 \\
.000\,000\,7 = .7 \times .08 = & 0^s.06 \\
\hline
 & 1^h\,13^m\,5^s.90
\end{array}$$

In practice the computer will perform the interpolation mentally, adding $.7 \times .08 = .06$ to the number 5.36 of the table in his head, and writing down $5^s.42$ as the last quantity to be added.

EXERCISES.

Convert into hours, minutes, and seconds:

1. $0^d.203\,079\,2$;
2. $0^d.783\,605\,8$;
3. $0^d.010\,203\,4$;
4. $0^d.990\,990\,9$.

To use the table for the reverse operation, we proceed as in the following example:

It is required to convert $17^h\ 29^m\ 30^s.93$ into decimals of a day. Looking in the table, we find that the required decimal is between 0.72 and 0.73. Hence the first two figures are 0.72, the equivalent of $17^h\ 16^m\ 48^s$. Subtracting the latter from the given number, we have a remainder $12^m\ 42^s.93$, to be sought for in column $\frac{H.M.S.}{100}$. This gives 88 as the next two figures. Subtracting the equivalent of .0088 or $12^m\ 40^s.32$, we have left $2^s.61$, which we are to seek in column $\frac{H.M.S.}{100^2}$. We find the corresponding number of column D to be 302. Hence

$$17^h\ 29^m\ 30^s.93 = 0^d.728\,830\,2.$$

$$\begin{array}{lcr}
 & & 17^h\ 29^m\ 30^s.93 \\
0.72 & = & 17^h\ 16^m\ 48^s \\
\hline
 & & 12^m\ 42^s.93 \\
.008\,8 & = & 12^m\ 40^s.32 \\
\hline
.000\,030\,2 & = & 2^s.61
\end{array}$$

In solving this problem the computer should be able, after a little practice, to perform the subtractions and carry the remainders mentally, thus saving himself the trouble of writing down the numbers.

EXERCISES.

Take the answers obtained from the four preceding exercises, subtract each result from $24^h\ 0^m\ 0^s$, change the remainder to decimals of a day, and see if when added to the decimals of the preceding exercises the sum is $1^d.000\,000\,0$, as it should be.

TABLE IX.

TO CONVERT TIME INTO ARC, AND *VICE VERSA*.

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23. In astronomy the right ascensions of the heavenly bodies are commonly given in hours, minutes, and seconds, the circumference being divided into 24 hours, each hour into 60 minutes, and each minute into 60 seconds.

Since $360^\circ =$ one circumference,

we have

$$1^h = 15^\circ;$$
$$1^m = 15';$$
$$1^s = 15'';$$

the signs ^{h}, ^{m}, and ^{s} indicating hours, minutes, and seconds of time.

Hence we may change time into arc by multiplying by 15, and arc into time by dividing by 15, the denominations being changed in each case. Table IX. enables us to do this by simple addition and subtraction by a process similar to that employed in changing hours, minutes, and seconds into decimals of a day.

To turn time into arc, we find in the table the whole number of degrees contained in the time denomination next smaller than the given one, and subtract the former time denomination from the latter.

Next we find the minutes of arc corresponding to the given time next smaller than the remainder, and again subtract.

Lastly we interpolate the seconds corresponding to the second remainder.

Example. Change $15^h\ 29^m\ 46^s.24$ to arc.

Given time, $15^h\ 29^m\ 46^s.24$
The table gives $232^\circ = 15^h\ 28^m$

Remainder, $1^m\ 46^s.24$
The table gives $26' =$ $1^m\ 44^s$

Remainder, $2^s.24 = 33''.6$

Hence

$$15^h\ 29^m\ 46^s.24 = 232^\circ\ 26'\ 33''.6.$$

The computer should be able to go through this operation without writing down anything but the result.

The operation of changing arc into time is too simple to require description, but it is more necessary to write down the work.

EXERCISES.

Change the following times to arc, and then check the results by changing the arcs into time and seeing whether the original times are reproduced:

1. $7^h\ 29^m\ 17^s.86$;
2. $0^h\ 4^m\ 0^s.25$;
3. $12^h\ 4^m\ 0^s.25$;
4. $13^h\ 48^m\ 16^s.40$;
5. $19^h\ 7^m\ 59^s.92$.

TABLE X.

TO CONVERT MEAN TIME INTO SIDEREAL TIME, AND SIDEREAL INTO MEAN TIME.

24. Since 365¼ solar days = 366¼ sidereal days (very nearly), any period expressed in mean time may be changed to sidereal time by increasing it by its $\frac{1}{365.25}$ part, and an interval of sidereal time may be changed to mean time by diminishing it by its $\frac{1}{366.25}$ part.

The first part of the table gives, for each 10 minutes of the argument, its $\frac{1}{365.25}$ part, by which it is to be increased. The second part of the table gives the $\frac{1}{366.25}$ part of the argument.

The small table in the margin shows the change for periods of less than 10 minutes.

Example 1. To change $17^h\ 48^m\ 36^s.7$ of mean time to sidereal time.

Given mean time,	$17^h\ 48^m\ 36^s.70$
Corr. for $17^h\ 40^m$,	$2^m\ 54^s.13$
Corr. for $8^m\ 37^s$,	$1^s.41$
Sidereal time,	$17^h\ 51^m\ 32^s.24$

Ex. 2. To change this interval of sidereal time back to mean time.

Corr. for $17^h\ 50^m$,	$-\ 2^m\ 55^s.29$
Corr. for $1^m\ 32^s$,	$-\ 0^s.25$
	$-\ 2^m\ 55^s.54$
Sidereal time,	$17^h\ 51^m\ 32^s.24$
Mean time,	$17^h\ 48^m\ 36^s.70$

EXERCISES.

Change to sidereal time:

1. $3^h\ 42^m\ 36^s.5$ m. t.; 3. $22^h\ 3^m\ 5^s.61$ m. t.
2. $18^h\ 46^m\ 29^s.82$ " 4. $0^h\ 1^m\ 12^s.55$ "

Change to mean time:

5. $0^h\ 7^m\ 16^s.3$ sidereal time;
6. $22^h\ 17^m\ 29^s.65$ "

OF DIFFERENCES AND INTERPOLATION.*

25. General Principles.

We call to mind that the object of a mathematical table is to enable one to find the value of a function corresponding to any value whatever of the variable argument. Since it is impossible to tabulate the function for all values of the argument, we have to construct the table for certain special values only, which values are generally equidistant. For example, in the tables of sines and cosines in the present work the values of the functions are given for values of the argument differing from each other by one minute.

The process of finding the values of functions corresponding to values of the argument intermediate between those given is called *interpolation.*

We have already had numerous examples of interpolation in its most simple form; we have now to consider the subject in a more general and extended way.

In the first place, we remark that, in strictness, no process of interpolation can be applicable to all cases whatever. From the mere facts that

To the number 2 corresponds the logarithm 0.301 03,
" " " 3 " " " 0.477 12,

we are not justified in drawing any conclusion whatever respecting the logarithms of numbers between 2 and 3. Hence some one or more hypotheses are always necessary as the base of any system of interpolation. The hypotheses always adopted are these two:

1. *That, supposing the argument to vary uniformly, the function varies according to some regular law.*

2. *That this law may be learned from the values of the function given in the table.*

These hypotheses are applied in the process of *differencing,* the nature of which will be seen by the following example, where it is applied to the logarithms of the numbers from 30 to 37:

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* The study of this subject will be facilitated by first mastering so much of it as is contained in the author's College Algebra, §§ 299–302.

It is also recommended to the beginner in the subject that, before going over the algebraic developments, he practise the methods of computation according to the rules and formulæ, so as to have a clear practical understanding of the notation. He can then more readily work out the developments.

	Function.	Δ'	Δ''	Δ'''	Δ^{iv}
log 30.	1.477 12				
		+ 1424			
" 31.	1.491 36		− 45		
		+ 1379		+ 2	
" 32.	1.505 15		− 43		+ 2
		+ 1336		+ 4	
" 33.	1.518 51		− 39		− 3
		+ 1297		+ 1	
" 34.	1.531 48		− 38		+ 1
		+ 1259		+ 2	
" 35.	1.544 07		− 36		+ 1
		+ 1223		+ 3	
" 36.	1.556 30		− 33		
		+ 1190			
" 37.	1.568 20				

The column Δ' gives each difference between two consecutive values of the function, formed by subtracting each number from that next following. These differences are called *first differences.*

The column Δ'' gives the difference between each two consecutive first differences. These are called *second differences.*

In like manner the numbers in the succeeding columns, when written, are called *third differences, fourth differences,* etc.

Now if, in continuing the successive orders of differences, we find them to continually become smaller and smaller, or to converge toward zero, this fact shows that the values of the functions follow a regular law, and the first hypothesis is therefore applicable.

In order to apply interpolation we must then assume that the intermediate values of the function follow the same law. The truth of this assumption must be established in some way before we can interpolate with mathematical rigor, but in practice we may suppose it true in the absence of any reason to the contrary.

26. *Effect of errors in the values of the functions.* In the preceding example it will be noticed that if we continue the orders of differences beyond the fourth, they will begin to increase and become irregular. This arises from the imperfections of the logarithms, owing to the omission of decimals beyond the fifth, already described in § 11.

When we find the differences to become thus irregular, we must be able to judge whether this irregularity arises from actual errors in the original numbers, which ought to be corrected, or from the small errors necessarily arising from the omission of decimals.

The great advantage of differencing is that any error, however small, in the quantities differenced, unless it follows a regular law, will be detected by the differences. To show the reason of this, we investigate what effect errors in the given functions will have upon the successive orders of differences.

Theorem. *The differences of the sum of two quantities are equal to the sums of their differences.*

General proof. Let

f_1, f_2, f_3, etc., be one set of functions;

f_1', f_2', f_3', etc., another set.

$f_1 + f_1', f_2 + f_2', f_3 + f_3'$, etc., will then be their sums.

In the first of the following columns we place the first differences of f, in the second those of f', and in the third those of $f + f'$, each formed according to the rule:

$$\begin{array}{ccc} f_2 - f_1 & f_2' - f_1' & f_2 + f_2' - (f_1 + f_1') \\ f_3 - f_2 & f_3' - f_2' & f_3 + f_3' - (f_2 + f_2') \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

It will be seen that the quantities in the third column are the sums of those in the first two.

Numerical Example.

f	Δ'	f'	Δ'	$f+f'$	Δ'
14		1		15	
	+ 25		+ 2		+ 27
39		3		42	
	+ 11		+ 3		+ 14
50		6		56	
	− 51		+ 4		− 47
− 1		10		9	

We see that the third set of values of Δ' follow the theorem.

Because the second differences are the differences of the first, the third the differences of the second, etc., it follows that the theorem is true for differences of any order.

Now when we write a series of functions in which the decimals exceeding a certain order are omitted, we may conceive each written number to be composed of the algebraic sum of two quantities, namely:

1. The true mathematical value of the function.
2. The negative of the omitted decimals.

Example. In the preceding collection of logarithms, since the true value of log 30 is 1.477 121 3 . . . , we may conceive the quantity written to be

$$1.477\,12 = \log 30 - .000\,001\,3 \ldots.$$

Hence the differences actually written are the differences of the true logarithms *minus* the differences of the errors. Now suppose the errors to be alternately + 0.5 and − 0.5 = the point marking off the last decimal. Their differences will then be as follows:

f'	Δ'	Δ''	Δ'''
− 0.5			
	+ 1		
+ 0.5		+ 2	
	− 1		− 4
− 0.5		− 2	
	+ 1		+ 4
+ 0.5		+ 2	
			− 4
		− 2	
etc.	etc.	etc.	etc.

It is evident that the nth order of differences of the errors are equal to $\pm 2^{n-1}$. Hence, in this case, if the nth order of differences of the true values of the function were zero, still, in consequence of the omission of decimals, the actual differences of the nth order would be 2^{n-1}.

This, however, is a very extreme case, since it is beyond all probability that the errors should alternate in this way. A more probable average example will be obtained by supposing a single number to have an error of 0.5, while the others are correct. We shall then have:

$$\begin{array}{cccccc}
f & \Delta' & \Delta'' & \Delta''' & \Delta^{iv} & \Delta^{v} \\
0 & & 0 & & +0.5 & \\
 & 0 & & +0.5 & & -2.5 \\
0 & & +0.5 & & -2.0 & \\
 & +0.5 & & -1.5 & & +5.0 \\
0.5 & & -1.0 & & +3.0 & \\
 & -0.5 & & +1.5 & & -5.0 \\
0 & & +0.5 & & -2.0 & \\
 & 0 & & -0.5 & & +2.5 \\
0 & & 0 & & +0.5 &
\end{array}$$

In this case the maximum value of the difference of the nth order is 1.5 in the differences of the third order, 3 in those of the fourth, 5 in those of the fifth, etc. Its general expression is

$$\frac{1}{2}\,\frac{n(n-1)(n-2)\ldots.(n-s+1)}{1.2.3\ldots.s},$$

where n is the order of differences, and

$$s = \frac{n}{2} \text{ or } \frac{n-1}{2}$$

according as n is even or odd. Thus:

$$\begin{aligned}
\Delta' &= \frac{1}{2}; \\
\Delta'' &= \frac{1}{2}\cdot\frac{2}{1} = 1; \\
\Delta''' &= \frac{1}{2}\cdot\frac{3}{1} = 1.5; \\
\Delta^{iv} &= \frac{1}{2}\cdot\frac{4.3}{1.2} = 3; \\
\Delta^{v} &= \frac{1}{2}\cdot\frac{5.4}{1.2} = 5; \\
\text{etc.} & \qquad \text{etc.}
\end{aligned}$$

This being about the average case, in actual practice the differences may be two or three times as great without necessarily implying an error greater than 0.5 in the numbers written.

We have now the following general rule for judging whether a series of numbers do really follow a uniform law:

Difference the series until we reach an order of differences in which the + and — signs either alternate or follow each other irregularly.

If none of the differences of this order expressed in units of the last place of decimals exceed the limit

$$\frac{n\,(n-1)\ldots\ldots(n-s+1)}{1\,.\,2\,.\,3\ldots\ldots s}$$

—that is, the value of the largest binomial coefficient of the nth order—the given numbers may be assumed to follow a regular law, and therefore to be correct to a unit in the last figure.

If some differences exceed this limit, their quotient by the above binomial coefficient may be considered to show the maximum error with which the number opposite it is probably affected.

We can thus detect an isolated error in a series of numbers with great certainty. Suppose, for example, an error of 2 in some number of the series. Differencing the series 0, 0, 0, 2, 0, 0, 0, we shall find the four largest differences of the fifth order to be − 10, + 20, − 20, + 10, which would enable us to hit at once upon the erroneous number and judge of the magnitude of its error.

An error near the beginning and end of the series of numbers of which the differences are taken cannot be detected by the differences unless it is considerable. If, for instance, the first or last number is in error by 1, the error of each order of differences will only be 1, as we may easily see by the following example:

$$\begin{array}{cccc} f' & \Delta' & \Delta'' & \Delta''' \\ 1 & & & \\ & -1 & & \\ 0 & & +1 & \\ & 0 & & -1 \text{ etc.} \\ 0 & & 0 & \end{array}$$

It is only in those differences which are on or near the same line as the numbers which are magnified in the way we have shown. But at the beginning and end of the series we cannot determine these differences.

Examining the various tables of differences, we see that n numbers have $n - 1$ first differences, $n - 2$ second differences, and so on, the number diminishing by 1 with each succeeding order. Hence, unless the number of given functions exceeds the index expressing the order of differences which we have to form, no certain conclusion can be drawn.

What is here said of the correctness of the numbers when the differences run properly must be understood as applicable to isolated errors only. If all the numbers were subject to an error following a regular law, this error would not be detected by the differences because, from the nature of the case, the latter only show deviations from some regular law.

27. Fundamental Formulæ of Interpolation.

We suppose a series of numbers to be differenced in the way already shown, and the various differences to be designated as in the following scheme, which is supposed to be a selection from a series preceding and following it.

Function.	1st Diff.	2d Diff.	3d Diff.	4th Diff.	5th Diff.
u_{-2}		Δ''_{-2}		Δ^{iv}_{-2}	
	$\Delta'_{-\frac{3}{2}}$		$\Delta'''_{-\frac{3}{2}}$		$\Delta^{v}_{-\frac{3}{2}}$
u_{-1}		Δ''_{-1}		Δ^{iv}_{-1}	
	$\Delta'_{-\frac{1}{2}}$		$\Delta'''_{-\frac{1}{2}}$		$\Delta^{v}_{-\frac{1}{2}}$
u_0		Δ''_0		Δ^{iv}_0	
	$\Delta'_{\frac{1}{2}}$		$\Delta'''_{\frac{1}{2}}$		$\Delta^{v}_{\frac{1}{2}}$
u_1		Δ''_1		Δ^{iv}_1	
	$\Delta'_{\frac{3}{2}}$		$\Delta'''_{\frac{3}{2}}$		$\Delta^{v}_{\frac{3}{2}}$
u_2		Δ''_2		Δ^{iv}_2	
	$\Delta'_{\frac{5}{2}}$		$\Delta'''_{\frac{5}{2}}$		$\Delta^{v}_{\frac{5}{2}}$
u_3		Δ''_3		Δ^{iv}_3	
	$\Delta'_{\frac{7}{2}}$				
u_4		Δ''_4			
etc.	etc.	etc.	etc.	etc.	etc.

It will be seen that the lower indices are chosen so as to show on which line a difference of any order falls. Thus all quantities with index 2 are on one horizontal line, those with index $\frac{5}{2} = 2\frac{1}{2}$ are half a line below, etc. This notation is a little different from that used in algebra, but the change need not cause any confusion.

It is shown in algebra that if n be any index, we have

$$u_n = u_0 + n\Delta'_{\frac{1}{2}} + \frac{n(n-1)}{1.2}\Delta''_1 + \frac{n(n-1)(n-2)}{1.2.3}\Delta'''_{\frac{3}{2}} + \text{etc.}; \qquad (a)$$

the notation being changed as in the preceding scheme.

Now the fundamental hypothesis of interpolation is that this formula, which can be demonstrated only for integral values of n, is true also for fractional values; that is, for values of the function u between those given in the table or in the above scheme. We therefore suppose this formula to express the value of the function u for any value of n between 0 and 1.

For values between + 1 and + 2 we have only to increase the indices of u and its differences by unity, thus:

$$u_{1+n} = u_1 + n\Delta'_{\frac{3}{2}} + \frac{n(n-1)}{1.2}\Delta''_2 + \text{etc,},$$

and by supposing n to increase from 0 to 1 in this formula we shall have values of u from u_1 to u_2.

Increasing the indices again—that is, applying our general formulæ to a row of quantities one line lower—we shall have

$$u_{2+n} = u_2 + n\Delta'_{\frac{5}{2}} + \frac{n\,(n-1)}{1\,.\,2}\Delta''_3 + \text{etc.}$$

The equation (a) is known as *Newton's formula of interpolation.*

28. Transformations of the Formula of Interpolation.

In the equation (a) and those following it, the formula of interpolation is not in its most convenient form. We shall therefore transform it so that the differences employed shall be symmetrical with respect to the functions between which the interpolation is to be made.

In working these transformations we shall suppose the sixth and following orders of differences to be so small as not to affect the result. These differences being supposed zero, any two consecutive fifth differences may be supposed equal.

First transformation. Let us first find what the original formula (a) will become when, instead of using the series of differences

$$\Delta'_{\frac{1}{2}},\quad \Delta''_1,\quad \Delta'''_{\frac{3}{2}},\quad \Delta^{\text{iv}}_2,\quad \text{etc.},$$

we use

$$\Delta'_{\frac{1}{2}},\quad \Delta''_0,\quad \Delta'''_{\frac{1}{2}},\quad \Delta^{\text{iv}}_0,\quad \text{etc.}$$

To effect the transformation we must find the values of the first series of differences in terms of the second, and substitute them in the formula (a).

We find, by the mode of forming the differences,

$$\begin{aligned}
\Delta''_1 &= \Delta''_0 + \Delta'''_{\frac{1}{2}};\\
\Delta'''_{\frac{3}{2}} &= \Delta'''_{\frac{1}{2}} + \Delta^{\text{iv}}_1;\\
&= \Delta'''_{\frac{1}{2}} + \Delta^{\text{iv}}_0 + \Delta^{\text{v}}_{\frac{1}{2}};\\
\Delta^{\text{iv}}_2 &= \Delta^{\text{iv}}_0 + \Delta^{\text{v}}_{\frac{1}{2}} + \Delta^{\text{v}}_{\frac{3}{2}};
\end{aligned}$$

for which, because we suppose the values of Δ^{v} to be equal, we may put

$$\begin{aligned}
\Delta^{\text{iv}}_2 &= \Delta^{\text{iv}}_0 + 2\Delta^{\text{v}}_{\frac{1}{2}};\\
\Delta^{\text{v}}_{\frac{3}{2}} &= \Delta^{\text{v}}_{\frac{1}{2}}.
\end{aligned}$$

Making these substitutions in (a), we have

$$\begin{aligned}
u_n = u_0 + n\Delta'_{\frac{1}{2}} &+ \frac{n\,(n-1)}{1\,.\,2}(\Delta''_0 + \Delta'''_{\frac{1}{2}})\\
&+ \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}(\Delta'''_{\frac{1}{2}} + \Delta^{\text{iv}}_0 + \Delta^{\text{v}}_{\frac{1}{2}})\\
&+ \frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,3\,.\,4}(\Delta^{\text{iv}}_0 + 2\Delta^{\text{v}}_{\frac{1}{2}})\\
&+ \frac{n\,(n-1)\,\ldots\,.\,(n-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\Delta^{\text{v}}_{\frac{1}{2}}.
\end{aligned}$$

Reducing by collecting the coefficients of equal differences, we find

$$u_n - u_0 = n\Delta'_{\frac{1}{2}} + \frac{n(n-1)}{1\,.\,2}\Delta_0'' + \frac{(n+1)\,n\,(n-1)}{1\,.\,2\,.\,3}\Delta'''_{\frac{1}{2}} + \frac{(n+1)\,n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3\,.\,4}\Delta^{\text{iv}}_0 + \frac{(n+2)(n+1)n(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\Delta^{\text{v}}_{\frac{1}{2}}. \qquad (b)$$

Second transformation. Next, instead of the series of this last formula, (*b*),

$$\Delta'_{\frac{1}{2}},\quad \Delta''_0,\quad \Delta'''_{\frac{1}{2}},\quad \Delta^{\text{iv}}_0,\quad \text{etc.,}$$

let us use

$$\Delta'_{-\frac{1}{2}},\quad \Delta''_0,\quad \Delta'''_{-\frac{1}{2}},\quad \Delta^{\text{iv}}_0,\quad \text{etc.}$$

To effect this transformation we substitute in (*b*) for $\Delta'_{\frac{1}{2}}$, $\Delta''_{\frac{1}{2}}$, etc.,

$$\begin{aligned} \Delta'_{\frac{1}{2}} &= \Delta'_{-\frac{1}{2}} + \Delta''_0; \\ \Delta'''_{\frac{1}{2}} &= \Delta'''_{-\frac{1}{2}} + \Delta^{\text{iv}}_0; \\ \Delta^{\text{v}}_{\frac{1}{2}} &= \Delta^{\text{v}}_{-\frac{1}{2}}. \end{aligned}$$

The series (*b*) then changes into

$$u_n - u_0 = n\Delta'_{-\frac{1}{2}} + \frac{n(n+1)}{1\,.\,2}\Delta''_0 + \frac{(n+1)\,n\,(n-1)}{1\,.\,2\,.\,3}\Delta'''_{-\frac{1}{2}} + \frac{(n+2)\,(n+1)\,n\,(n-1)}{1\,.\,2\,.\,3\,.\,4}\Delta^{\text{iv}}_0 + \frac{(n+2)(n+1)\,n\,(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\Delta^{\text{v}}_{-\frac{1}{2}}. \qquad (c)$$

Third transformation. Stirling's formula. We effect a third transformation by taking the half sum of the equations (*b*) and (*c*), which gives us a formula perfectly symmetrical with respect to the lines of differences, namely,

$$u_n - u_0 = n\frac{\Delta'_{-\frac{1}{2}} + \Delta'_{\frac{1}{2}}}{2} + \frac{n^2}{2}\Delta''_0 + \frac{n(n^2-1)}{1.2.3}\,\frac{\Delta'''_{-\frac{1}{2}} + \Delta'''_{\frac{1}{2}}}{2} + \frac{n^2(n^2-1)}{1.2.3.4}\Delta^{\text{iv}}_0 + \frac{n(n^2-1)(n^2-4)}{1.2.3.4.5}\,\frac{\Delta^{\text{v}}_{-\frac{1}{2}} + \Delta^{\text{v}}_{\frac{1}{2}}}{2} + \text{etc.,} \qquad (d)$$

which is known as *Stirling's formula of interpolation.*

It will be seen that we have put

$n^2 - 1$ for $(n+1)(n-1)$,
$n^2 - 4$ for $(n+2)(n-2)$,
etc. etc.

Fourth transformation. In the equation (*b*), instead of the series of differences

$$\Delta'_{\frac{1}{2}},\quad \Delta''_0,\quad \Delta'''_{\frac{1}{2}},\quad \Delta^{\text{iv}}_0,\quad \text{etc.,}$$

let us use

$$\Delta'_{\frac{1}{2}},\quad \Delta''_1,\quad \Delta'''_{\frac{1}{2}},\quad \Delta^{\text{iv}}_1,\quad \text{etc.}$$

To effect this we put

$$\Delta''_0 = \Delta''_1 - \Delta'''_{\frac{1}{2}};$$
$$\Delta^{\text{iv}}_0 = \Delta^{\text{iv}}_1 - \Delta^{\text{v}}_{\frac{1}{2}}.$$

Making these substitutions in (*b*), it becomes

$$u_n - u_0 = n\Delta'_{\frac{1}{2}} + \frac{n(n-1)}{1.2}\Delta''_1 + \frac{n(n-1)(n-2)}{1.2.3}\Delta'''_{\frac{1}{2}}$$
$$+ \frac{(n+1)n(n-1)(n-2)}{1.2.3.4}\Delta^{\text{iv}}_1$$
$$+ \frac{(n+1)n(n-1)(n-2)(n-3)}{1.2.3.4.5}\Delta^{\text{v}}_{\frac{1}{2}}. \qquad (e)$$

Fifth transformation. Bessel's formula. Let us take half the sum of the equations (*e*) and (*b*). We then have

$$u_n - u_0 = n\Delta'_{\frac{1}{2}} + \frac{n(n-1)}{1.2}\,\frac{\Delta''_0 + \Delta''_1}{2} + \frac{n(n-1)(n-\frac{1}{2})}{1.2.3}\Delta'''_{\frac{1}{2}}$$
$$+ \frac{(n+1)n(n-1)(n-2)}{1.2.3.4}\,\frac{\Delta^{\text{iv}}_0 + \Delta^{\text{iv}}_1}{2}$$
$$+ \frac{(n+1)n(n-1)(n-2)(n-\frac{1}{2})}{1.2.3.4.5}\Delta^{\text{v}}_{\frac{1}{2}}, \qquad (f)$$

which is commonly known as *Bessel's formula of interpolation*, and which is the one most convenient to use in practice.

In applying this formula to find a value of the function intermediate between two given values, we must always suppose the index 0 to apply to the given value next preceding that to be found, and the index 1 to apply to that next following. The quantity n will then be a positive proper fraction.

29. *Example of interpolation to halves.* If we increase the logarithms of 30, 31, etc., already given, by unity, we shall have the logarithms of 300, 310, 320, etc. It is required to find, by interpolation, the logarithms of the numbers half way between the given ones (omitting the first and last), namely, the logarithms of 315, 325, 335, etc.

Here, the required quantities depending upon arguments half way between the given ones, we have $n = \frac{1}{2}$, and the values of the Besselian coefficient, so far as wanted, are

$$\frac{n(n-1)}{2} = -\frac{1}{8};$$
$$\frac{n(n-1)(n-\frac{1}{2})}{6} = 0.$$

The subsequent terms are neglected, being insensible. So, if we put a_0 and a_1 for any consecutive two of the numbers 300, 310, etc., we have

$$\left.\begin{aligned} \log(a_0+5) &= \log a_0 + \left(\frac{1}{2}\Delta'_{\frac{1}{2}} - \frac{1}{8}\,\frac{\Delta''_0+\Delta''_1}{2}\right) \\ \text{and}\qquad \log(a_1-5) &= \log a_1 - \left(\frac{1}{2}\Delta'_{\frac{1}{2}} + \frac{1}{8}\,\frac{\Delta''_0+\Delta''_1}{2}\right), \end{aligned}\right\} \qquad (h)$$

where we put $\Delta_{\frac{1}{2}}$ for that first difference between a_0 and a_1.

These two formulæ are two expressions for the same quantity, because $a_0 + 5 = a_1 - 5$. They are both used in such a way as to provide a check upon the accuracy of the work. For this purpose we compute the two quantities

$$\left.\begin{aligned} \log(a_0+5) - \log a_0 &= \frac{1}{2}\Delta'_{\frac{1}{2}} - \frac{1}{8}\,\frac{\Delta''_0+\Delta''_1}{2}, \\ \log a_1 - \log(a_0+5) &= \frac{1}{2}\Delta'_{\frac{1}{2}} + \frac{1}{8}\,\frac{\Delta''_0+\Delta''_1}{2}. \end{aligned}\right\} \qquad (k)$$

The most convenient and expeditious way of doing the work is shown in the accompanying table, where we give every figure which it is necessary to write, besides those found on p. 57. The following is the plan of computation:

No.	Log.	Diff.	$\frac{1}{2}\Delta'_{\frac{1}{2}}$.	$\frac{1}{8}\frac{\Delta''_0+\Delta''_1}{2}$.	$\frac{\Delta''_0+\Delta''_1}{2}$.
310	2.491 36				
		695			
315	.498 31		+ 689.5	− 5.5	− 44
		684			
320	.505 15				
		673			
325	.511 88		668.0	− 5.1	− 41
		663			
330	.518 51				
		653			
335	.525 04		648.5	− 4.8	− 38
		644			
340	.531 48				
		634			
345	.537 82		629.5	− 4.6	− 37
		625			
350	.544 07				
		616			
355	.550 23		+ 611.5	− 4.3	− 34
		607			
360	2.556 30				

We compute the right-hand column by the formula

$$\frac{\Delta''_0+\Delta''_1}{2} = \Delta''_0 + \frac{1}{2}\Delta'''_{\frac{1}{2}} = \Delta''_1 - \frac{1}{2}\Delta'''_{\frac{1}{2}},$$

using the values of Δ given in the scheme, p. 57.

This mode of computing the half sum of two numbers which are nearly equal is easier than adding and dividing by 2.

In the next two columns to the left, the sixth place of decimals is added in order that the errors may not accumulate by the addition of several quantities. This precaution should always be taken when the interpolated quantities are required to be as accurate as the given ones.

The fourth column from the right is formed by adding and subtracting the numbers of the second and third columns according to the formula (k). The additional figure is now dropped, because no longer necessary for accuracy. The numbers thus formed are the first differences of the series of logarithms found by inserting the interpolated logarithms between the given ones, as will be seen by equation (k).

We write the first logarithm of the series, namely,

$$\log 310 = 2.491\,36,$$

and then form the subsequent ones by continual addition of the differences, thus:

$$\begin{aligned} \log 315 &= \log 310 + 695; \\ \log 320 &= \log 315 + 684; \\ \log 325 &= \log 320 + 673; \\ \text{etc.} \quad & \quad \text{etc.} \quad \text{etc.} \end{aligned}$$

If the work is correct, the alternate logarithms will agree with the given ones in the former table.

The continuance of the above process for a few more numbers, say up to 450, is recommended to the student as an exercise.

30. *Interpolation to thirds.* Let us suppose the value of a quantity to be given for every third day, and the value for every day to be required. By putting $n = \frac{1}{3}$ and applying formula (f) to each successive given quantity, we shall have the value for each day following one of those given, and by putting $n = \frac{2}{3}$ we shall have values for the second day following, which will complete the series. But the interpolation can be executed by a much more expeditious process, which consists in computing the middle difference of the interpolated quantities and finding the intermediate differences by a secondary interpolation.

Let us put

f_0, f_3, f_6, etc., the given series of quantities;

f_0, f_1, f_2, f_3, f_4, etc., the required interpolated series;

Δ', Δ'', etc., the first differences, second differences, etc., of the given series;

δ', δ'', etc., the first differences, second differences, etc., of the interpolated series.

We may then put

$$f_3 - f_0 = \Delta'_{\frac{1}{2}} \quad \text{(in the given series);}$$

$$\left.\begin{aligned} f_1 - f_0 &= \delta'_{\frac{1}{6}} \\ f_2 - f_1 &= \delta'_{\frac{1}{2}} \\ f_3 - f_2 &= \delta'_{\frac{5}{6}} \end{aligned}\right\} \text{(in the interpolated series).}$$

We shall then have

$$\delta'_{\frac{1}{6}} + \delta'_{\frac{1}{2}} + \delta'_{\frac{5}{6}} = \Delta'_{\frac{1}{2}}.$$

The value of $f_1 - f_0 = \delta_{\frac{1}{6}}$ is given by putting $n = \frac{1}{3}$ in the Besselian formula (f). Thus we find

$$\delta'_{\frac{1}{6}} = \frac{1}{3}\Delta'_{\frac{1}{2}} - \frac{1}{9}\,\frac{\Delta''_0 + \Delta''_1}{2} + \frac{1}{162}\Delta'''_{\frac{1}{2}} + \frac{5}{243}\,\frac{\Delta^{\text{iv}}_0 + \Delta^{\text{iv}}_1}{2} - \frac{1}{1458}\,\Delta^{\text{v}}_{\frac{1}{2}}.$$

Putting $n = \frac{2}{3}$, we have the value of $f_2 - f_0$, that is, of $\delta'_{\frac{1}{6}} + \delta'_{\frac{1}{2}}$. Thus we find

$$\delta'_{\frac{1}{6}} + \delta'_{\frac{1}{2}} = \frac{2}{3}\Delta'_{\frac{1}{2}} - \frac{1}{9}\,\frac{\Delta''_0 + \Delta''_1}{2} - \frac{1}{162}\,\Delta'''_{\frac{1}{2}} + \frac{5}{243}\,\frac{\Delta^{\text{iv}}_0 + \Delta^{\text{iv}}_1}{2} + \frac{1}{1458}\,\Delta^{\text{v}}_{\frac{1}{2}}.$$

Subtracting these expressions, we have

$$\delta'_{\frac{1}{2}} = \frac{1}{3}\,\Delta'_{\frac{1}{2}} - \frac{1}{81}\,\Delta'''_{\frac{1}{2}} + \frac{1}{729}\,\Delta^{\text{v}}_{\frac{1}{2}},$$

which is most easily computed in the form

$$\delta'_{\frac{1}{2}} = \frac{1}{3}\left\{\Delta'_{\frac{1}{2}} - \frac{1}{27}\left(\Delta'''_{\frac{1}{2}} - \frac{1}{9}\,\Delta^{\text{v}}_{\frac{1}{2}}\right)\right\}. \tag{m}$$

We see that the computation of $\delta'_{\frac{1}{2}}$, the middle difference of the interpolated quantities, is much simpler than that of $\delta'_{\frac{1}{6}}$. It will therefore facilitate the work to compute only these middle differences, and to find the others by interpolation.

This process is again facilitated, in case the second differences are considerable, by first computing the second differences of the interpolated series on the same plan. The formulæ for this purpose are derived as follows:

Let us put

$$\delta'_{\frac{7}{6}} = f_4 - f_3.$$

The second difference of which we desire the value is then

$$\delta''_3 = \delta'_{\frac{7}{6}} - \delta'_{\frac{5}{6}}.$$

The value of $\delta'_{\frac{5}{6}}$ is given by the equation

$$\delta'_{\frac{5}{6}} = \Delta'_{\frac{1}{2}} - (\delta'_{\frac{1}{6}} + \delta'_{\frac{1}{2}}),$$

and the value of $\delta'_{\frac{3}{2}}$ is found from that of $\delta'_{\frac{1}{2}}$ by simply increasing the indices of the differences by unity, because it belongs to the next lower line.

We thus find

$$\delta'_{\frac{3}{2}} = \frac{1}{3}\Delta'_{\frac{3}{2}} - \frac{1}{9}\,\frac{\Delta''_1 + \Delta''_2}{2} + \frac{1}{162}\Delta'''_{\frac{3}{2}} + \frac{5}{243}\,\frac{\Delta^{\mathrm{iv}}_1 + \Delta^{\mathrm{iv}}_2}{2} - \frac{1}{1458}\Delta^{\mathrm{v}}_{\frac{3}{2}};$$

$$\delta'_{\frac{1}{2}} = \frac{1}{3}\Delta'_{\frac{1}{2}} + \frac{1}{9}\,\frac{\Delta''_0 + \Delta''_1}{2} + \frac{1}{162}\Delta'''_{\frac{1}{2}} - \frac{5}{243}\,\frac{\Delta^{\mathrm{iv}}_0 + \Delta^{\mathrm{iv}}_1}{2} - \frac{1}{1458}\Delta^{\mathrm{v}}_{\frac{1}{2}}.$$

Then by subtraction,

$$\delta''_2 = \frac{1}{3}(\Delta'_{\frac{3}{2}} - \Delta'_{\frac{1}{2}}) - \frac{1}{9}\,\frac{\Delta''_0 + 2\Delta''_1 + \Delta''_2}{2} + \frac{1}{162}(\Delta'''_{\frac{3}{2}} - \Delta'''_{\frac{1}{2}}) + \frac{5}{243}\,\frac{\Delta^{\mathrm{iv}}_0 + 2\Delta^{\mathrm{iv}}_1 + \Delta''_2}{2} - \frac{1}{1458}(\Delta^{\mathrm{v}}_{\frac{3}{2}} - \Delta^{\mathrm{v}}_{\frac{1}{2}}).$$

Reducing the first of these terms, we have

$$\Delta'_{\frac{3}{2}} - \Delta'_{\frac{1}{2}} = \Delta''_1.$$

For the second term,

$$\Delta''_0 = \Delta''_1 - \Delta'''_{\frac{1}{2}};$$
$$\Delta''_2 = \Delta''_1 + \Delta'''_{\frac{3}{2}};$$

whence

$$\Delta''_0 + \Delta''_2 = 2\Delta''_1 + \Delta'''_{\frac{3}{2}} - \Delta'''_{\frac{1}{2}} = 2\Delta''_1 + \Delta^{\mathrm{iv}}_1,$$

and

$$\frac{\Delta''_0 + 2\Delta''_1 + \Delta''_2}{2} = 2\Delta''_1 + \frac{1}{2}\Delta^{\mathrm{iv}}_1.$$

For the third term,

$$\Delta'''_{\frac{3}{2}} - \Delta'''_{\frac{1}{2}} = \Delta^{\mathrm{iv}}_1.$$

For the fourth term, dropping the terms in Δ^{vi} as too small in practice, we may put

$$\frac{\Delta^{\mathrm{iv}}_0 + 2\Delta^{\mathrm{iv}}_1 + \Delta^{\mathrm{iv}}_2}{2} = 2\Delta^{\mathrm{iv}}_1.$$

The difference of the fifth terms may also be dropped, because they contain only sixth differences.

Making these substitutions in the value of δ''_2, we find

$$\begin{aligned}\delta''_2 &= \frac{1}{3}\Delta''_1 - \frac{1}{9}\left(2\Delta''_1 + \frac{1}{2}\Delta^{\mathrm{iv}}_1\right) + \frac{1}{162}\Delta^{\mathrm{iv}}_1 + \frac{10}{243}\Delta^{\mathrm{iv}}_1\\ &= \frac{1}{9}\Delta''_1 - \frac{2}{243}\Delta^{\mathrm{iv}}_1\\ &= \frac{1}{9}\left(\Delta''_1 - \frac{2}{27}\Delta^{\mathrm{iv}}\right). \qquad (n)\end{aligned}$$

By this formula we may compute every third value of δ'', and then interpolate the intermediate values. By means of these values we find by addition the intermediate values of δ', of which every third value has been computed by formula (m). Then, by continually adding the values of δ', we find those of the function f.

As an example of the work, we give the following values of the sun's declination for every third day of part of July, 1886, for Greenwich mean noon:

Date.	☉'s Dec.			Δ'		Δ''	Δ'''
1886.	°	′	″	′	″	″	″
July 3......	22	57	37.5				
				– 16	28.3		
6......	22	41	9.2			– 212.4	
				– 20	0.7		+ 4.5
9......	22	21	8.5			– 207.9	
				– 23	28.6		+ 4.5
12......	21	57	39.9			– 203.4	
				– 26	52.0		+ 5.7
15......	21	30	47.9			– 197.7	
				– 30	9.7		
18......	21	0	38.2				

The values of Δ^{iv} are too small to have any influence.

The whole work of interpolation is shown in the following table, where, as before, the right-hand column is that first computed, and gives the value of $\Delta' - \frac{1}{27}\Delta'''$ according to formula (m):

Date.	☉'s Dec.			δ'		δ''	$\Delta' - \frac{1}{27}\Delta'''$	
1886.	°	′	″	′	″	″		
July **6**......	**22**	**41**	**9.2**			**– 23.60**		
				– 6	16.86			
7......	22	34	52.4			– 23.43		
				– 6	**40.29**		– 20	0.87
8......	22	28	12.1			– 23.27		
				– 7	3.56			
9......	**22**	**21**	**8.5**			**– 23.10**		
				– 7	26.66			
10......	22	13	41.9			– 22.93		
				– 7	**49.59**		– 23	28.77
11......	22	5	52.3			– 22.78		
				– 8	12.37			
12......	**21**	**57**	**39.9**			**– 22.61**		
				– 8	34.98			
13......	21	49	4.9			– 22.42		
				– 8	**57.40**		– 26	52.21
14......	21	40	7.5			– 22.19		
				– 9	19.59			
15......	**21**	**30**	**47.9**			**– 21.19**		

To make the process in the example clear, the computed differences, etc., are printed in heavier type than the interpolated ones.

It is also to be remarked that the sum of the three consecutive values of δ'', formed of one computed value and the interpolated values next above and below it, should be equal to the difference between the corresponding computed first differences. For instance,

$$23''.27 + 23''.10 + 22''.93 = 7'\ 49''.59 - 6'\ 40''.29.$$

But in the first computation this condition will seldom be exactly fulfilled, owing to the errors arising from omitted decimals and other sources. If the given quantities are accurate, the errors should never exceed half a unit of the last decimal in the given quantities, or five units in the additional decimal added on in dividing.

To correct these little imperfections after the interpolation of the second differences, but before that of the first differences, the sum of the last two figures in each triplet of second differences should be formed, and if it does not agree with the difference of the first differences, the last figures of the second difference should each be slightly altered, to make the sum exact.

The first differences can then be formed by addition.

In the same way, the sum of three consecutive first differences should be equal to the difference between the given quantities. If, as is generally the case, this condition is not exactly fulfilled, the differences should be altered accordingly. This alteration may, however, be made mentally while adding to form the required interpolated functions.

As an exercise for the student we give the continuance of the sun's declination for the remainder of the month, to be interpolated for the intermediate dates from July 15th onward:

	°	′	″
July 21	20	27	16.5
24	19	50	49.1
27	19	11	22.7
30	18	29	4.8
Aug. 2	17	44	3.1

As another exercise the logarithms of the intermediate numbers from 998 to 1014 may be interpolated by the following table:

Number.	Logarithm.
994	2.997 386 4
997	2.998 695 2
1000	3.000 000 0
1003	3.001 300 9
1006	3.002 598 0
1009	3.003 891 2
1012	3.005 180 5
1015	3.006 466 0
1018	3.007 747 8

32. *Interpolation to fifths.* Let us next investigate the formulæ when every fifth quantity is given and the intermediate ones are to be found by interpolation. By putting $n = \frac{2}{5}$ in the Besselian formula, we shall have the value of the interpolation function second following one of the given ones, and by putting $n = \frac{3}{5}$ that third following. The difference will be the middle interpolated first difference of the interpolated series. Putting $n = \frac{2}{5}$ in (f), we have

$$u_{\frac{2}{5}} = u_0 - \frac{2}{5}\Delta'_{\frac{1}{2}} - \frac{2.3}{2.5^2}\,\frac{\Delta''_0 + \Delta''_1}{2} + \frac{2.3.1}{2^2.3.5^3}\Delta'''_{\frac{1}{2}}$$
$$+ \frac{2.3.7{:}8}{2.3.4.5^4}\,\frac{\Delta^{\text{iv}}_0 + \Delta^{\text{iv}}_1}{2} - \frac{2.3.7.8.1}{2^2.3.4.5.5^5}\Delta^{\text{v}}_{\frac{1}{2}}.$$

Putting $n = \frac{3}{5}$, we have

$$u_{\frac{3}{5}} = u_0 + \frac{3}{5}\Delta'_{\frac{1}{2}} - \frac{2.3}{2.5^2}\,\frac{\Delta''_0 + \Delta''_1}{2} - \frac{2.3.1}{2^2.3.5^3}\Delta'''_{\frac{1}{2}}$$
$$+ \frac{2.3.7.8}{2.3.4.5^4}\,\frac{\Delta^{\text{iv}}_0 + \Delta^{\text{iv}}_1}{2} + \frac{8.3.2.7.1}{2^2.3.4.5.5^5}\Delta^{\text{v}}_{\frac{1}{2}}.$$

The difference of these expressions, being reduced, gives

$$u_{\frac{3}{5}} - u_{\frac{2}{5}} = \frac{1}{5}\Delta'_{\frac{1}{2}} - \frac{1}{125}\Delta'''_{\frac{1}{2}} + \frac{14}{15625}\Delta^{\text{v}}_{\frac{1}{2}}$$
$$= \frac{1}{5}\left\{\Delta'_{\frac{1}{2}} - \frac{1}{25}\left(\Delta'''_{\frac{1}{2}} - \frac{14}{125}\Delta^{\text{v}}_{\frac{1}{2}}\right)\right\}.$$

The term in Δ^{v} will not produce any effect unless the fifth differences are considerable, and then we may nearly always, in practice, put $\frac{1}{9}$ instead of $\frac{14}{125}$.

The interpolated second differences opposite the given functions are most readily obtained by Stirling's formula, (d). Putting $n = \frac{1}{5}$, we have the following value of the interpolated first differences immediately following a given value of the function:

$$u_{\frac{1}{5}} - u_0 = \frac{1}{5}\,\frac{\Delta'_{-\frac{1}{2}} + \Delta'_{\frac{1}{2}}}{2} + \frac{1}{50}\Delta''_0 + \frac{24}{6.5.25}\,\frac{\Delta'''_{-\frac{1}{2}} + \Delta'''_{\frac{1}{2}}}{2}$$
$$+ \frac{24}{6.5.20.25}\Delta^{\text{iv}}_0 + \text{etc.}$$

Again, putting $n = -\frac{1}{5}$, and changing the signs, we find for the first difference next preceding a given function

$$u_0 - u_{-\frac{1}{5}} = \frac{1}{5}\,\frac{\Delta'_{-\frac{1}{2}} + \Delta'_{\frac{1}{2}}}{2} - \frac{1}{50}\Delta''_0 + \frac{24}{6.5.25}\,\frac{\Delta'''_{-\frac{1}{2}} + \Delta'''_{\frac{1}{2}}}{2}$$
$$- \frac{24}{6.5.20.25}\Delta^{\text{iv}}_0 - \text{etc.}$$

The difference of these quantities gives the required second difference, which we find to be

$$\delta''_0 = \frac{1}{25}\Delta''_0 - \frac{2}{625}\Delta^{\text{iv}}_0 = \frac{1}{25}\left(\Delta''_0 - \frac{2}{25}\Delta^{\text{iv}}_0\right).$$

As an example and exercise we show the interpolation of logarithms when every fifth logarithm is given:

Number.	Logarithm.	δ'	δ''	Δ'	Δ''
1000	3.000 000 0				
				+ 21 661	
1005	**3.002 166 1**		**− 4.32**		− 108
		4319.2			
1006	.002 598 0		− 4.31		
		4314.9			
1007	.003 029 5		− 4.30		
		4310.6			
1008	.003 460 6		− 4.30	+ 21 553	
		4306.3			
1009	.003 891 2		− 4.29		
		4302.0			
1010	**3.004 321 4**		**− 4.28**		− 107
		4297.7			
1011	.004 751 2		− 4.27		
		4293.5			
1012	.005 180 5		− 4.26		
		4289.2			
1013	.005 609 4		− 4.23	+ 21 446	
		4285.0			
1014	.006 037 9		− 4.20		
		4280.8			
1015	**3.006 466 0**		**− 4.16**		− 104
				+ 21 342	
1020	3.008 600 2				
1025	3.010 723 9				
1030	3.012 837 2				
1035	3.014 940 3				
1040	3.017 033 3				

FORMULÆ

FOR THE SOLUTION OF

PLAIN AND SPHERICAL TRIANGLES.

REMARKS.

1. It is better to determine an angle by its tangent than by its sine or cosine, because a small angle or an angle near 180° cannot be accurately determined by its cosine, nor one near either 90° or 270° by its sine.

Sometimes, however, the data of the problem are such that the angle can be determined only through its sine or cosine. Any uncertainty which may then arise from the source pointed out is then inherent in the problem; e.g., if the hypothenuse and one side of a right triangle are 0.39808 and 0.39806 respectively (sixth and following decimals being omitted), the value of the included angle may be anywhere between 0° 25′ and 0° 42′, no matter what method of computation be adopted.

2. If the sine and cosine can be independently computed, their agreement as to the angle will generally serve as a check on the accuracy of the computation. If they agree, their quotient will give the tangent.

3. It is desirable, when possible, to have a check upon the accuracy of the computation; that is, to make a computation which must give a certain result if the work is right. But no check can give a positive assurance of accuracy: all it can do is to make it more or less improbable that a mistake exceeding a certain limit exists.

4. In the following list several formulæ are sometimes given as applicable to the same problem. In such cases, the most convenient for the special purpose must be chosen.

Notation. a, b, and c are the three sides.
A, B, and C are the opposite angles.

PLANE TRIANGLES.

Given.	Required.	
$a, b, c,$ the three sides.	A, one angle.	$s = \frac{1}{2}(a+b+c).$ $\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$
	$A, B, C,$ all the angles.	$H = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$ $\tan\frac{1}{2}A = \frac{H}{s-a};$ $\tan\frac{1}{2}B = \frac{H}{s-b};$ $\tan\frac{1}{2}C = \frac{H}{s-c}.$ Checks: $A + B + C = 180°;$ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
$b, c, A,$ two sides and the included angle.	B and C, the other angles.	$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{1}{2}A;$ $\frac{1}{2}(B+C) = 90° - \frac{1}{2}A;$ $B = \frac{1}{2}(B+C) + \frac{1}{2}(B-C);$ $C = \frac{1}{2}(B+C) - \frac{1}{2}(B-C).$ Check, as before.
	$a, B, C,$ the remaining parts.	$a\sin\frac{1}{2}(B-C) = (b-c)\cos\frac{1}{2}A;$ $a\cos\frac{1}{2}(B-C) = (b+c)\sin\frac{1}{2}A.$ Having found a and $\frac{1}{2}(B-C)$, proceed as in the last case.
$a, b, A,$ two sides and the angle opposite one of them.	$c, B, C,$ the remaining parts.	$\sin B = \frac{b}{a}\sin A;$ (two values of B.) $C = 180° - (A+B);$ $c = \frac{b\sin C}{\sin B} = \frac{a\sin C}{\sin A}.$

Given.	Required.	
$a, A, B,$ one side and any two angles.	$b, c, C,$ the remaining parts.	$C = 180° - (A + B)$; $b = \frac{a \sin B}{\sin A}$; $c = \frac{a \sin C}{\sin A} = \frac{a \sin (A + B)}{\sin A}$.

RIGHT SPHERICAL TRIANGLES.

		c is the hypothenuse.
$a, b,$ the sides containing the right angle.	$A, B,$ or c.	$\cot A = \cot a \sin b$; $\cot B = \cot b \sin a$; $\cos c = \cos a \cos b$; $\sin c = \frac{\sin a}{\sin A}$.
	A and c.	$\sin c \sin A = \sin a$; $\sin c \cos A = \cos a \sin b$; $\cos c = \cos a \cos b$.
	B and c	$\sin c \sin B = \sin b$; $\sin c \cos B = \sin a \cos b$.
$a, c,$ one side and the hypothenuse.	$A, B,$ or b.	$\sin A = \frac{\sin a}{\sin c}$; $\cos B = \tan a \cot c$; $\cos b = \frac{\cos c}{\cos a}$.
$a, A,$ one side and the opposite angle.	$b, c,$ or B.	$\sin b = \tan a \cot A$; $\sin c = \frac{\sin a}{\sin A}$; $\sin B = \frac{\cos A}{\cos a}$.
$a, B,$ one side and the adjacent angle.	$b, c,$ or A.	$\tan b = \sin a \tan B$; $\tan c = \frac{\tan a}{\cos B}$; $\cos A = \cos a \sin B$.
	c and A.	$\sin A \sin c = \sin a$; $\sin A \cos c = \cos a \cos B$; $\cos A = \cos a \sin B$.

Given.	**Required.**	
a, B.	b and A.	$\sin A \sin b = \sin a \sin B$; $\sin A \cos b = \cos B$.
c, A, the hypothenuse and one angle.	a, b, or B.	$\sin a = \sin c \sin A$; $\tan b = \tan c \cos A$; $\cot B = \cos c \tan A$.
	a and B.	$\cos a \sin B = \cos A$; $\cos a \cos B = \sin A \cos c$; $\sin a = \sin A \sin c$.
	a and b.	$\cos a \sin b = \cos A \sin c$; $\cos a \cos b = \cos c$.
A, B, the two angles.	a, b, or c.	$\cos a = \frac{\cos A}{\sin B}$; $\cos b = \frac{\cos B}{\sin A}$; $\cos c = \cot A \cot B$.

QUADRANTAL SPHERICAL TRIANGLES.

		c is the omitted side equal to 90°. C is the angle opposite this side.
a, b, the two sides.	A, B, or C, either angle.	$\cos A = \frac{\cos a}{\sin b}$; $\cos B = \frac{\cos b}{\sin a}$; $\cos C = -\cot a \cot b$.
a, C, one side and the angle opposite the right side.	A, B, or b.	$\sin A = \sin a \sin C$; $\tan B = -\cos a \tan C$; $\cot b = -\tan a \cos C$.
	A and b.	$\cos A \sin b = \cos a$; $\cos A \cos b = -\sin a \cos C$. $\sin A = \sin a \sin C$.
	A and B.	$\cos A \sin B = \cos a \sin C$; $\cos A \cos B = -\cos C$.

Given.	Required.	
A, b, one angle and the adjacent side.	a, B, or C.	$\cos a = \cos A \sin b$; $\tan B = \sin A \tan b$; $\cot C = -\cot A \cos b$.
	a and B.	$\sin a \sin B = \sin A \sin b$; $\sin a \cos B = \cos b$; $\cos a = \cos A \sin b$.
	a and C.	$\sin a \sin C = \sin A$; $\sin a \cos C = -\cos A \cos b$.
a, A, one side and the opposite angle.	b, B, or C.	$\sin b = \frac{\cos a}{\cos A}$; $\sin B = \cot a \tan A$; $\sin C = \frac{\sin A}{\sin a}$.
A, C, one angle and the angle opposite the right side.	a, b, or B.	$\sin a = \frac{\sin A}{\sin C}$; $\cos b = -\tan A \cot C$; $\cos B = -\frac{\cos C}{\cos A}$.
A, B, two angles.	a, b, or C.	$\cot a = \cot A \sin B$; $\cot b = \sin A \cot B$; $\cos C = -\cos A \cos B$.
	a and C.	$\sin C \sin a = \sin A$; $\sin C \cos a = \cos A \sin B$; $\cos C = -\cos A \cos B$.
	b and C.	$\sin C \sin b = \sin B$; $\sin C \cos b = \sin A \cos B$.

SPHERICAL TRIANGLES IN GENERAL.

Given.	Required.	
$a, b, c,$ the three sides.	$A, B, C,$ the three angles.	$s = \frac{1}{2}(a + b + c);$ $H = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$ $\tan\frac{1}{2}A = \frac{H}{\sin(s-a)};$ $\tan\frac{1}{2}B = \frac{H}{\sin(s-b)};$ $\tan\frac{1}{2}C = \frac{H}{\sin(s-c)}.$ Check: $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$
$a, b, C,$ two sides and the included angle.	A and $c,$ one angle and the remaining side.	$\sin c \sin A = \sin a \sin C;$ $\sin c \cos A = \cos a \sin b - \sin a \cos b \cos C;$ $\cos c = \cos a \cos b + \sin a \sin b \cos C.$
	B and $c.$	$\sin c \sin B = \sin b \sin C;$ $\sin c \cos B = \sin a \cos b - \cos a \sin b \cos C.$ If addition and subtraction logarithms are not available for this computation, we may compute k and K from $k \sin K = \sin a \cos C;$ $k \cos K = \cos a.$ Then $\sin c \cos A = k \sin(b - K);$ $\cos c = k \cos(b - K).$ Also, $h \sin H = \sin b \cos C;$ $h \cos H = \cos b.$ Then $\sin c \cos B = h \sin(a - H);$ $\cos c = h \cos(a - H).$
	$A, B, c,$ all the remaining parts.	$\sin\frac{1}{2}c \sin\frac{1}{2}(A - B) = \cos\frac{1}{2}C \sin\frac{1}{2}(a - b);$ $\sin\frac{1}{2}c \cos\frac{1}{2}(A - B) = \sin\frac{1}{2}C \sin\frac{1}{2}(a + b);$ $\cos\frac{1}{2}c \sin\frac{1}{2}(A + B) = \cos\frac{1}{2}C \cos\frac{1}{2}(a - b);$ $\cos\frac{1}{2}c \cos\frac{1}{2}(A + B) = \sin\frac{1}{2}C \cos\frac{1}{2}(a + b).$

Given.	Required.	
a, b, A, two sides and an opposite angle.	B, C, c, all the remaining parts.	$\sin B = \frac{\sin A \sin b}{\sin a}$ (two values of B); $\tan \frac{1}{2} C = \frac{\cos \frac{1}{2}(a-b) \cot \frac{1}{2}(A+B)}{\cos \frac{1}{2}(a+b)}$; $\tan \frac{1}{2} c = \frac{\cos \frac{1}{2}(A+B) \tan \frac{1}{2}(a+b)}{\cos \frac{1}{2}(A-B)}$.
A, B, c, two angles and the included side.	a and C, one side and the third angle.	$\sin C \sin a = \sin A \sin c$; $\sin C \cos a = \cos A \sin B + \sin A \cos B \cos c$; $\cos C = -\cos A \cos B + \sin A \sin B \cos c$.
	b and C.	$\sin C \sin b = \sin B \sin c$; $\sin C \cos b = \sin A \cos B + \cos A \sin B \cos c$. If we compute k and K from $k \sin K = \cos A$, $k \cos K = \sin A \cos c$, then $\sin C \cos a = k \cos (B - K)$; $\cos C = k \sin (B - K)$. If we compute h and H from $h \sin H = \cos B$, $h \cos H = \sin B \cos c$, then $\sin C \cos b = h \cos (A - H)$; $\cos C = h \sin (A - H)$.
	a, b, C, all the remaining parts.	$\sin \frac{1}{2} C \sin \frac{1}{2}(a+b) = \sin \frac{1}{2} c \cos \frac{1}{2}(A-B)$; $\sin \frac{1}{2} C \cos \frac{1}{2}(a+b) = \cos \frac{1}{2} c \cos \frac{1}{2}(A+B)$; $\cos \frac{1}{2} C \sin \frac{1}{2}(a-b) = \sin \frac{1}{2} c \sin \frac{1}{2}(A-B)$; $\cos \frac{1}{2} C \cos \frac{1}{2}(a-b) = \cos \frac{1}{2} c \sin \frac{1}{2}(A+B)$.
A, B, a, two angles and an opposite side.	b, c, C, all the remaining parts.	$\sin b = \frac{\sin a \sin B}{\sin A}$ (two values of b); $\tan \frac{1}{2} c = \frac{\cos \frac{1}{2}(A+B) \tan \frac{1}{2}(a+b)}{\cos \frac{1}{2}(A-B)}$; $\tan \frac{1}{2} C = \frac{\cos \frac{1}{2}(a-b) \cot \frac{1}{2}(A+B)}{\cos \frac{1}{2}(a+b)}$.
A, B, C, the three angles.	a, b, c, the three sides.	$S = \frac{1}{2}(A+B+C)$; $P = \sqrt{\frac{-\cos S}{\cos (S-A) \cos (S-B) \cos (S-C)}}$; $\tan \frac{1}{2} a = P \cos (S - A)$; $\tan \frac{1}{2} b = P \cos (S - B)$; $\tan \frac{1}{2} c = P \cos (S - C)$.

TABLES.

TABLE I.

COMMON LOGARITHMS

OF NUMBERS.

N.	Log.	N.	Log.	N.	Log.	N.	Log.	N.	Log.
0	— Infinity.	**30**	1.47 712	**60**	1.77 815	**90**	1.95 424	**120**	2.07 918
1	0.00 000	31	1.49 136	61	1.78 533	91	1.95 904	121	2.08 279
2	0.30 103	32	1.50 515̄	62	1.79 239	92	1.96 379	122	2.08 636
3	0.47 712	33	1.51 851	63	1.79 934	93	1.96 848	123	2.08 991
4	0.60 206	34	1.53 148	64	1.80 618	94	1.97 313	124	2.09 342
5	0.69 897	35	1.54 407	65	1.81 291	95	1.97 772	125	2.09 691
6	0.77 815	36	1.55 630	66	1.81 954	96	1.98 227	126	2.10 037
7	0.84 510	37	1.56 820	67	1.82 607	97	1.98 677	127	2.10 380
8	0.90 309	38	1.57 978	68	1.83 251	98	1.99 123	128	2.10 721
9	0.95 424	39	1.59 106	69	1.83 885̄	99	1.99 564	129	2.11 059
10	1.00 000	**40**	1.60 206	**70**	1.84 510	**100**	2.00 000	**130**	2.11 394
11	1.04 139	41	1.61 278	71	1.85 126	101	2.00 432	131	2.11 727
12	1.07 918	42	1.62 325̄	72	1.85 733	102	2.00 860	132	2.12 057
13	1.11 394	43	1.63 347	73	1.86 332	103	2.01 284	133	2.12 385
14	1.14 613	44	1.64 345	74	1.86 923	104	2.01 703	134	2.12 710
15	1.17 609	45	1.65 321	75	1.87 506	105	2.02 119	135	2.13 033
16	1.20 412	46	1.66 276	76	1.88 081	106	2.02 531	136	2.13 354
17	1.23 045̄	47	1.67 210	77	1.88 649	107	2.02 938	137	2.13 672
18	1.25 527	48	1.68 124	78	1.89 209	108	2.03 342	138	2.13 988
19	1.27 875	49	1.69 020	79	1.89 763	109	2.03 743	139	2.14 301
20	1.30 103	**50**	1.69 897	**80**	1.90 309	**110**	2.04 139	**140**	2.14 613
21	1.32 222	51	1.70 757	81	1.90 849	111	2.04 532	141	2.14 922
22	1.34 242	52	1.71 600	82	1.91 381	112	2.04 922	142	2.15 229
23	1.36 173	53	1.72 428	83	1.91 908	113	2.05 308	143	2.15 534
24	1.38 021	54	1.73 239	84	1.92 428	114	2.05 690	144	2.15 836
25	1.39 794	55	1.74 036	85	1.92 942	115	2.06 070	145	2.16 137
26	1.41 497	56	1.74 819	86	1.93 45̄0	116	2.06 446	146	2.16 435
27	1.43 136	57	1.75 587	87	1.93 952	117	2.06 819	147	2.16 732
28	1.44 716	58	1 76 343	88	1.94 448	118	2.07 188	148	2.17 026
29	1.46 240	59	1.77 085	89	1.94 939	119	2.07 555	149	2.17 319
30	1.47 712	**60**	1.77 815	**90**	1.95 424	**120**	2.07 918	**150**	2.17 609

TABLE I.

N.	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
01	432	475	518	561	604	647	689	732	775̅	817
02	860	903	945	988	*030	*072	*115̅	*157	*199	*242
03	01 284	326	368	410	452	494	536	578	620	662
04	703	745	787	828	870	912	953	995̅	*036	*078
05	02 119	160	202	243	284	325	366	407	449	490
06	531	572	612	653	694	735̅	776	816	857	898
07	938	979	*019	*060	*100	*141	*181	*222	*262	*302
08	03 342	383	423	463	503	543	583	623	663	703
09	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415̅	454	493
11	532	571	610	65̅0	689	727	766	805	844	883
12	922	961	999	*038	*077	*115	*154	*192	*231	*269
13	05 308	346	385̅	423	461	5̅00	538	576	614	652
14	690	729	767	805̅	843	881	918	956	994	*032
15	06 070	108	145	183	221	258	296	333	371	408
16	446	483	521	558	595	633	670	707	744	781
17	819	856	893	930	967	*004	*041	*078	*115̅	*151
18	07 188	225̅	262	298	335	372	408	445	482	518
19	555̅	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135̅	*171	*207	*243
21	08 279	314	350	386	422	458	493	529	565̅	600
22	636	672	707	743	778	814	849	884	920	955
23	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
24	09 342	377	412	447	482	517	552	587	621	656
25	691	726	760	795	830	864	899	934	968	*003
26	10 037	072	106	140	175̅	209	243	278	312	346
27	380	415̅	449	483	517	551	585	619	653	687
28	721	755̅	789	823	857	890	924	958	992	*025
29	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
31	727	760	793	826	860	893	926	959	992	*024
32	12 057	090	123	156	189	222	254	287	320	352
33	385	418	450	483	516	548	581	613	646	678
34	710	743	775	808	840	872	905̅	937	969	*001
35	13 033	066	098	130	162	194	226	258	290	322
36	354	386	418	45̅0	481	513	545	577	609	640
37	672	704	735	767	799	830	862	893	925̅	956
38	988	*019	*051	*082	*114	*145̅	*176	*208	*239	*270
39	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675̅	706	737	768	799	829	860	891
41	922	953	983	*014	*045̅	*076	*106	*137	*168	*198
42	15 229	259	290	320	351	381	412	442	473	503
43	534	564	594	625̅	655̅	685	715	746	776	806
44	836	866	897	927	957	987	*017	*047	*077	*107
45	16 137	167	197	227	256	286	316	346	376	406
46	435	465	495̅	524	554	584	613	643	673	702
47	732	761	791	820	85̅0	879	909	938	967	997
48	17 026	056	085̅	114	143	173	202	231	260	289
49	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725̅	754	782	811	840	869
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Prop. Pts.

N.		0	1	2	3	4	5	6	7	8	9
150	17	609	638	667	696	725̄	754	782	811	840	869
51		898	926	955	984	*013	*041	*070	*099	*127	*156
52	18	184	213	241	270	298	327	355	384	412	441
53		469	498	526	554	583	611	639	667	696	724
54		752	780	808	837	865̄	893	921	949	977	*005
55	19	033	061	089	117	145	173	201	229	257	285̄
56		312	340	368	396	424	451	479	507	535̄	562
57		590	618	645	673	700	728	756	783	811	838
58		866	893	921	948	976	*003	*030	*058	*085	*112
59	20	140	167	194	222	249	276	303	330	358	385̄
160		412	439	466	493	520	548	575̄	602	629	656
61		683	710	737	763	790	817	844	871	898	925̄
62		952	978	*005	*032	*059	*085	*112	*139	*165	*192
63	21	219	245	272	299	325	352	378	405̄	431	458
64		484	511	537	564	590	617	643	669	696	722
65		748	775̄	801	827	854	880	906	932	958	985̄
66	22	011	037	063	089	115	141	167	194	220	246
67		272	298	324	35̄0	376	401	427	453	479	505
68		531	557	583	608	634	660	686	712	737	763
69		789	814	840	866	891	917	943	968	994	*019
170	23	045̄	070	096	121	147	172	198	223	249	274
71		300	325	350	376	401	426	452	477	502	528
72		553	578	603	629	654	679	704	729	754	779
73		805̄	830	855̄	880	905̄	930	955̄	980	*005̄	*030
74	24	055̄	080	105̄	130	155̄	180	204	229	254	279
75		304	329	353	378	403	428	452	477	502	527
76		551	576	601	625	65̄0	674	699	724	748	773
77		797	822	846	871	895	920	944	969	993	*018
78	25	042	066	091	115	139	164	188	212	237	261
79		285	310	334	358	382	406	431	455̄	479	503
180		527	551	575	600	624	648	672	696	720	744
81		768	792	816	840	864	888	912	935	959	983
82	26	007	031	055̄	079	102	126	150	174	198	221
83		245	269	293	316	340	364	387	411	435̄	458
84		482	505	529	553	576	600	623	647	670	694
85		717	741	764	788	811	834	858	881	905̄	928
86		951	975̄	998	*021	*045̄	*068	*091	*114	*138	*161
87	27	184	207	231	254	277	300	323	346	370	393
88		416	439	462	485	508	531	554	577	600	623
89		646	669	692	715	738	761	784	807	830	852
190		875	898	921	944	967	989	*012	*035	*058	*081
91	28	103	126	149	171	194	217	240	262	285̄	307
92		330	353	375	398	421	443	466	488	511	533
93		556	578	601	623	646	668	691	713	735	758
94		780	803	825̄	847	870	892	914	937	959	981
95	29	003	026	048	070	092	115̄	137	159	181	203
96		226	248	270	292	314	336	358	380	403	425̄
97		447	469	491	513	535̄	557	579	601	623	645̄
98		667	688	710	732	754	776	798	820	842	863
99		885	907	929	951	973	994	*016	*038	*060	*081
200	30	103	125̄	146	168	190	211	233	255̄	276	298
N.		0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

Prop. Pts.

TABLE I.

N.	0	1	2	3	4	5	6	7	8	9
200	30 103	125̄	146	168	190	211	233	255̄	276	298
01	320	341	363	384	406	428	449	471	492	514
02	535	557	578	600	621	643	664	685	707	728
03	75̄0	771	792	814	835	856	878	899	920	942
04	963	984	*006	*027	*048	*069	*091	*112	*133	*154
05	31 175	197	218	239	260	281	302	323	345̄	366
06	387	408	429	45̄0	471	492	513	534	555	576
07	597	618	639	660	681	702	723	744	765̄	785
08	806	827	848	869	890	911	931	952	973	994
09	32 015̄	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305̄	325	346	366	387	408
11	428	449	469	490	510	531	552	572	593	613
12	634	654	675̄	695̄	715	736	756	777	797	818
13	838	858	879	899	919	940	960	980	*001	*021
14	33 041	062	082	102	122	143	163	183	203	224
15	244	264	284	304	325̄	345̄	365̄	385	405	425
16	445	465	486	506	526	546	566	586	606	626
17	646	666	686	706	726	746	766	786	806	826
18	846	866	885	905	925	945	965	985̄	*005̄	*025̄
19	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
21	439	459	479	498	518	537	557	577	596	616
22	635	655̄	674	694	713	733	753	772	792	811
23	830	85̄0	869	889	908	928	947	967	986	*005
24	35 025̄	044	064	083	102	122	141	160	180	199
25	218	238	257	276	295	315̄	334	353	372	392
26	411	430	449	468	488	507	526	545	564	583
27	603	622	641	660	679	698	717	736	755	774
28	793	813	832	851	870	889	908	927	946	965̄
29	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305̄	324	342
31	361	380	399	418	436	455	474	493	511	530
32	549	568	586	605̄	624	642	661	680	698	717
33	736	754	773	791	810	829	847	866	884	903
34	922	940	959	977	996	*014	*033	*051	*070	*088
35	37 107	125	144	162	181	199	218	236	254	273
36	291	310	328	346	365̄	383	401	420	438	457
37	475̄	493	511	530	548	566	585̄	603	621	639
38	658	676	694	712	731	749	767	785	803	822
39	840	858	876	894	912	931	949	967	985̄	*003
240	38 021	039	057	075	093	112	130	148	166	184
41	202	220	238	256	274	292	310	328	346	364
42	382	399	417	435	453	471	489	507	525̄	543
43	561	578	596	614	632	65̄0	668	686	703	721
44	739	757	775̄	792	810	828	846	863	881	899
45	917	934	952	970	987	*005	*023	*041	*058	*076
46	39 094	111	129	146	164	182	199	217	235̄	252
47	270	287	305̄	322	340	358	375	393	410	428
48	445	463	480	498	515	533	550	568	585	602
49	620	637	655̄	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Pts.

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
51	967	98$\bar{5}$	*002	*019	*037	*054	*071	*088	*106	*123
52	40 140	157	17$\bar{5}$	192	209	226	243	261	278	29$\bar{5}$
53	312	329	346	364	381	398	41$\bar{5}$	432	449	466
54	483	500	518	53$\bar{5}$	552	569	586	603	620	637
55	654	671	688	705	722	739	756	773	790	807
56	824	841	858	87$\bar{5}$	892	909	926	943	960	976
57	993	*010	*027	*044	*061	*078	*09$\bar{5}$	*111	*128	*145
58	41 162	179	196	212	229	246	263	280	296	313
59	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
61	664	681	697	714	731	747	764	780	797	814
62	830	847	863	880	896	913	929	946	963	979
63	996	*012	*029	*045	*062	*078	*09$\bar{5}$	*111	*127	*144
64	42 160	177	193	210	226	243	259	275	292	308
65	32$\bar{5}$	341	357	374	390	406	423	439	455	472
66	488	504	521	537	553	570	586	602	619	63$\bar{5}$
67	651	667	684	700	716	732	749	76$\bar{5}$	781	797
68	813	830	846	862	878	894	911	927	943	959
69	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	18$\bar{5}$	201	217	233	249	26$\bar{5}$	281
71	297	313	329	34$\bar{5}$	361	377	393	409	42$\bar{5}$	441
72	457	473	489	50$\bar{5}$	521	537	553	569	584	600
73	616	632	648	664	680	696	712	727	743	759
74	775	791	807	823	838	854	870	886	902	917
75	933	949	96$\bar{5}$	981	996	*012	*028	*044	*059	*075
76	44 091	107	122	138	154	170	185	201	217	232
77	248	264	279	29$\bar{5}$	311	326	342	358	373	389
78	404	420	436	451	467	483	498	514	529	54$\bar{5}$
79	560	576	592	607	623	638	654	669	68$\bar{5}$	700
280	716	731	747	762	778	793	809	824	840	855
81	871	886	902	917	932	948	963	979	994	*010
82	45 02$\bar{5}$	040	056	071	086	102	117	133	148	163
83	179	194	209	22$\bar{5}$	240	255	271	286	301	317
84	332	347	362	378	393	408	423	439	454	469
85	484	$\bar{5}$00	51$\bar{5}$	530	545	561	576	591	606	621
86	637	652	667	682	697	712	728	743	758	773
87	788	803	818	834	849	864	879	894	909	924
88	939	954	969	984	*000	*01$\bar{5}$	*030	*04$\bar{5}$	*060	*07$\bar{5}$
89	46 090	10$\bar{5}$	120	13$\bar{5}$	1$\bar{5}$0	16$\bar{5}$	180	19$\bar{5}$	210	22$\bar{5}$
290	240	25$\bar{5}$	270	28$\bar{5}$	300	31$\bar{5}$	330	34$\bar{5}$	359	374
91	389	404	419	434	449	464	479	494	509	523
92	538	553	568	583	598	613	627	642	657	672
93	687	702	716	731	746	761	776	790	805	820
94	83$\bar{5}$	8$\bar{5}$0	864	879	894	909	923	938	953	967
95	982	997	*012	*026	*041	*056	*070	*085	*100	*114
96	47 129	144	159	173	188	202	217	232	246	261
97	276	290	30$\bar{5}$	319	334	349	363	378	392	407
98	422	436	451	465	480	494	509	524	538	553
99	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	18	17	16	15	14
1	1.8	1.7	1.6	1.5	1.4
2	3.6	3.4	3.2	3.0	2.8
3	5.4	5.1	4.8	4.5	4.2
4	7.2	6.8	6.4	6.0	5.6
5	9.0	8.5	8.0	7.5	7.0
6	10.8	10.2	9.6	9.0	8.4
7	12.6	11.9	11.2	10.5	9.8
8	14.4	13.6	12.8	12.0	11.2
9	16.2	15.3	14.4	13.5	12.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
800	47 712	727	741	756	770	784	799	813	828	842
01	857	871	885	900	914	929	943	958	972	986
02	48 001	015	029	044	058	073	087	101	116	130
03	144	159	173	187	202	216	230	244	259	273
04	287	302	316	330	344	359	373	387	401	416
05	430	444	458	473	487	501	515	530	544	558
06	572	586	601	61$\bar{5}$	629	643	657	671	686	700
07	714	728	742	756	770	78$\bar{5}$	799	813	827	841
08	855	869	883	897	911	926	940	954	968	982
09	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
810	49 136	150	164	178	192	206	220	234	248	262
11	276	290	304	318	332	346	360	374	388	402
12	415	429	443	457	471	485	499	513	527	541
13	554	568	582	596	610	624	638	651	665	679
14	693	707	721	734	748	762	776	790	803	817
15	831	84$\bar{5}$	859	872	886	900	914	927	941	95$\bar{5}$
16	969	982	996	*010	*024	*037	*051	*06$\bar{5}$	*079	*092
17	50 106	120	133	147	161	174	188	202	215	229
18	243	256	270	284	297	311	32$\bar{5}$	338	352	365
19	379	393	406	420	433	447	461	474	488	501
820	51$\bar{5}$	529	542	556	569	583	596	610	623	637
21	651	664	678	691	70$\bar{5}$	718	732	745	759	772
22	786	799	813	826	840	853	866	880	893	907
23	920	934	947	961	974	987	*001	*014	*028	*041
24	51 05$\bar{5}$	068	081	09$\bar{5}$	108	121	13$\bar{5}$	148	162	17$\bar{5}$
25	188	202	215	228	242	255	268	282	295	308
26	322	335	348	362	375	388	402	41$\bar{5}$	428	441
27	45$\bar{5}$	468	481	49$\bar{5}$	508	521	534	548	561	574
28	587	601	614	627	640	654	667	680	693	706
29	720	733	746	759	772	786	799	812	825	838
830	851	86$\bar{5}$	878	891	904	917	930	943	957	970
31	983	996	*009	*022	*035	*048	*061	*07$\bar{5}$	*088	*101
32	52 114	127	140	153	166	179	192	205	218	231
33	244	257	270	284	297	310	323	336	349	362
34	37$\bar{5}$	388	401	414	427	440	453	466	479	492
35	504	517	530	543	556	569	582	595	608	621
36	634	647	660	673	686	699	711	724	737	750
37	763	776	789	802	81$\bar{5}$	827	840	853	866	879
38	892	90$\bar{5}$	917	930	943	956	969	982	994	*007
39	53 020	033	046	058	071	084	097	110	122	135
840	148	161	173	186	199	212	224	237	2$\bar{5}$0	263
41	275	288	301	314	326	339	352	364	377	390
42	403	415	428	441	453	466	479	491	504	517
43	529	542	55$\bar{5}$	567	580	593	605	618	631	643
44	656	668	681	694	706	719	732	744	757	769
45	782	794	807	820	832	84$\bar{5}$	857	870	882	895
46	908	920	933	945	958	970	983	995	*008	*020
47	54 033	045	058	070	083	095	108	120	133	145
48	158	170	183	195	208	220	233	245	258	270
49	283	29$\bar{5}$	307	320	332	34$\bar{5}$	357	370	382	394
850	407	419	432	444	456	469	481	494	506	518
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
51	531	543	555	568	580	593	605̄	617	630	642
52	654	667	679	691	704	716	728	741	753	765
53	777	790	802	814	827	839	851	864	876	888
54	900	913	925̄	937	949	962	974	986	998	*011
55	55 023	035	047	060	072	084	096	108	121	133
56	145̄	157	169	182	194	206	218	230	242	255̄
57	267	279	291	303	315	328	340	352	364	376
58	388	400	413	425̄	437	449	461	473	485	497
59	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715̄	727	739
61	751	763	775̄	787	799	811	823	835̄	847	859
62	871	883	895̄	907	919	931	943	955̄	967	979
63	991	*003	*015̄	*027	*038	*050	*062	*074	*086	*098
64	56 110	122	134	146	158	170	182	194	205	217
65	229	241	253	265̄	277	289	301	312	324	336
66	348	360	372	384	396	407	419	431	443	455̄
67	467	478	490	502	514	526	538	549	561	573
68	585̄	597	608	620	632	644	656	667	679	691
69	703	714	726	738	750	761	773	785̄	797	808
370	820	832	844	855	867	879	891	902	914	926
71	937	949	961	972	984	996	*008	*019	*031	*043
72	57 054	066	078	089	101	113	124	136	148	159
73	171	183	194	206	217	229	241	252	264	276
74	287	299	310	322	334	345	357	368	380	392
75	403	415̄	426	438	449	461	473	484	496	507
76	519	530	542	553	565̄	576	588	600	611	623
77	634	646	657	669	680	692	703	715̄	726	738
78	749	761	772	784	795	807	818	830	841	852
79	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
81	58 092	104	115	127	138	149	161	172	184	195̄
82	206	218	229	240	252	263	274	286	297	309
83	320	331	343	354	365	377	388	399	410	422
84	433	444	456	467	478	490	501	512	524	535̄
85	546	557	569	580	591	602	614	625̄	636	647
86	659	670	681	692	704	715̄	726	737	749	760
87	771	782	794	805̄	816	827	838	850	861	872
88	883	894	906	917	928	939	950	961	973	984
89	995̄	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
91	218	229	240	251	262	273	284	295	306	318
92	329	340	351	362	373	384	395	406	417	428
93	439	450	461	472	483	494	506	517	528	539
94	550	561	572	583	594	605̄	616	627	638	649
95	660	671	682	693	704	715̄	726	737	748	759
96	770	780	791	802	813	824	835	846	857	868
97	879	890	901	912	923	934	945̄	956	966	977
98	988	999	*010	*021	*032	*043	*054	*065̄	*076	*086
99	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

Prop. Pts.

TABLE I.

N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
400	60 206	217	228	239	249	260	271	282	293	304	
01	314	325	336	347	358	369	379	390	401	412	
02	423	433	444	455	466	477	487	498	509	520	
03	531	541	552	563	574	584	595	606	617	627	
04	638	649	660	670	681	692	703	713	724	735̄	
05	746	756	767	778	788	799	810	821	831	842	
06	853	863	874	885̄	895	906	917	927	938	949	11
07	959	970	981	991	*002	*013	*023	*034	*045̄	*055	1 1.1
08	61 066	077	087	098	109	119	130	140	151	162	2 2.2
09	172	183	194	204	215̄	225	236	247	257	268	3 3.3
410	278	289	300	310	321	331	342	352	363	374	4 4.4
11	384	395̄	405	416	426	437	448	458	469	479	5 5.5
12	490	500	511	521	532	542	553	563	574	584	6 6.6
13	595	606	616	627	637	648	658	669	679	690	7 7.7
14	700	711	721	731	742	752	763	773	784	794	8 8.8
15	805̄	815	826	836	847	857	868	878	888	899	9 9.9
16	909	920	930	941	951	962	972	982	993	*003	
17	62 014	024	034	045̄	055	066	076	086	097	107	
18	118	128	138	149	159	170	180	190	201	211	
19	221	232	242	252	263	273	284	294	304	315̄	
420	325̄	335	346	356	366	377	387	397	408	418	
21	428	439	449	459	469	480	490	500	511	521	10
22	531	542	552	562	572	583	593	603	613	624	1 1.0
23	634	644	655̄	665̄	675	685	696	706	716	726	2 2.0
24	737	747	757	767	778	788	798	808	818	829	3 3.0
25	839	849	859	870	880	890	900	910	921	931	4 4.0.
26	941	951	961	972	982	992	*002	*012	*022	*033	5 5.0
27	63 043	053	063	073	083	094	104	114	124	134	6 6.0
28	144	155̄	165̄	175̄	185̄	195	205	215	225	236	7 7.0
29	246	256	266	276	286	296	306	317	327	337	8 8.0
430	347	357	367	377	387	397	407	417	428	438	9 9.0
31	448	458	468	478	488	498	508	518	528	538	
32	548	558	568	579	589	599	609	619	629	639	
33	649	659	669	679	689	699	709	719	729	739	
34	749	759	769	779	789	799	809	819	829	839	
35	849	859	869	879	889	899	909	919	929	939	
36	949	959	969	979	988	998	*008	*018	*028	*038	9
37	64 048	058	068	078	088	098	108	118	128	137	1 0.9
38	147	157	167	177	187	197	207	217	227	237	2 1.8
39	246	256	266	276	286	296	306	316	326	335	3 2.7
440	345	355	365	375̄	385̄	395̄	404	414	424	434	4 3.6
41	444	454	464	473	483	493	503	513	523	532	5 4.5
42	542	552	562	572	582	591	601	611	621	631	6 5.4
43	640	650	660	670	680	689	699	[illegible]09	719	729	7 6.3
44	738	748	758	768	777	787	797	807	816	826	8 7.2
45	836	846	856	865	875	885̄	895̄	904	914	924	9 8.1
46	933	943	953	963	972	982	992	*002	*011	*021	
47	65 031	040	050	060	070	079	089	099	108	118	
48	128	137	147	157	167	176	186	196	205	215̄	
49	225̄	234	244	254	263	273	283	292	302	312	
450	321	331	341	350	360	369	379	389	398	408	
N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.

N.		0	1	2	3	4	5	6	7	8	9
450	65	321	331	341	350	360	369	379	389	398	408
51		418	427	437	447	456	466	475	485	495̄	504
52		514	523	533	543	552	562	571	581	591	600
53		610	619	629	639	648	658	667	677	686	696
54		706	715	725̄	734	744	753	763	772	782	792
55		801	811	820	830	839	849	858	868	877	887
56		896	906	916	925	935̄	944	954	963	973	982
57		992	*001	*011	*020	*030	*039	*049	*058	*068	*077
58	66	087	096	106	115̄	124	134	143	153	162	172
59		181	191	200	210	219	229	238	247	257	266
460		276	285	295̄	304	314	323	332	342	351	361
61		370	380	389	398	408	417	427	436	445	455̄
62		464	474	483	492	502	511	521	530	539	549
63		558	567	577	586	596	605̄	614	624	633	642
64		652	661	671	680	689	699	708	717	727	736
65		745	755̄	764	773	783	792	801	811	820	829
66		839	848	857	867	876	885	894	904	913	922
67		932	941	950	960	969	978	987	997	*006	*015
68	67	025̄	034	043	052	062	071	080	089	099	108
69		117	127	136	145	154	164	173	182	191	201
470		210	219	228	237	247	256	265	274	284	293
71		302	311	321	330	339	348	357	367	376	385̄
72		394	403	413	422	431	440	449	459	468	477
73		486	495	504	514	523	532	541	550	560	569
74		578	587	596	605	614	624	633	642	651	660
75		669	679	688	697	706	715	724	733	742	752
76		761	770	779	788	797	806	815	825̄	834	843
77		852	861	870	879	888	897	906	916	925̄	934
78		943	952	961	970	979	988	997	*006	*015	*024
79	68	034	043	052	061	070	079	088	097	106	115
480		124	133	142	151	160	169	178	187	196	205
81		215̄	224	233	242	251	260	269	278	287	296
82		305̄	314	323	332	341	350	359	368	377	386
83		395̄	404	413	422	431	440	449	458	467	476
84		485̄	494	502	511	520	529	538	547	556	565̄
85		574	583	592	601	610	619	628	637	646	655̄
86		664	673	681	690	699	708	717	726	735	744
87		753	762	771	780	789	797	806	815	824	833
88		842	851	860	869	878	886	895	904	913	922
89		931	940	949	958	966	975	984	993	*002	*011
490	69	020	028	037	046	055	064	073	082	090	099
91		108	117	126	135̄	144	152	161	170	179	188
92		197	205	214	223	232	241	249	258	267	276
93		285̄	294	302	311	320	329	338	346	355	364
94		373	381	390	399	408	417	425	434	443	452
95		461	469	478	487	496	504	513	522	531	539
96		548	557	566	574	583	592	601	609	618	627
97		636	644	653	662	671	679	688	697	705	714
98		723	732	740	749	758	767	775	784	793	801
99		810	819	827	836	845̄	854	862	871	880	888
500		897	906	914	923	932	940	949	958	966	975
N.		0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
01	984	992	*001	*010	*018	*027	*036	*044	*053	*062
02	70 070	079	088	096	10$\bar{5}$	114	122	131	140	148
03	157	165	174	183	191	200	209	217	226	234
04	243	252	260	269	278	286	29$\bar{5}$	303	312	321
05	329	338	346	35$\bar{5}$	364	372	381	389	398	406
06	415	424	432	441	449	458	467	475	484	492
07	501	509	518	526	535	544	552	561	569	578
08	586	59$\bar{5}$	603	612	621	629	638	646	65$\bar{5}$	663
09	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
11	842	851	859	868	876	88$\bar{5}$	893	902	910	919
12	927	935	944	952	961	969	978	986	99$\bar{5}$	*003
13	71 012	020	029	037	046	054	063	071	079	088
14	096	10$\bar{5}$	113	122	130	139	147	155	164	172
15	181	189	198	206	214	223	231	240	248	257
16	26$\bar{5}$	273	282	290	299	307	315	324	332	341
17	349	357	366	374	383	391	399	408	416	42$\bar{5}$
18	433	441	4$\bar{5}$0	458	466	47$\bar{5}$	483	492	$\bar{5}$00	508
19	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
21	684	692	700	709	717	725	734	742	750	759
22	767	775	784	792	800	809	817	825	834	842
23	850	858	867	875	883	892	900	908	917	92$\bar{5}$
24	933	941	9$\bar{5}$0	958	966	97$\bar{5}$	983	991	999	*008
25	72 016	024	032	041	049	057	066	074	082	090
26	099	107	115	123	132	140	148	156	16$\bar{5}$	173
27	181	189	198	206	214	222	230	239	247	255
28	263	272	280	288	296	304	313	321	329	337
29	346	354	362	370	378	387	39$\bar{5}$	403	411	419
530	428	436	444	452	460	469	477	48$\bar{5}$	493	501
31	509	518	526	534	542	550	558	567	57$\bar{5}$	583
32	591	599	607	616	624	632	640	648	656	66$\bar{5}$
33	673	681	689	697	705	713	722	730	738	746
34	754	762	770	779	787	79$\bar{5}$	803	811	819	827
35	835	843	852	860	868	876	884	892	900	908
36	916	92$\bar{5}$	933	941	949	957	965	973	981	989
37	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
38	73 078	086	094	102	111	119	127	13$\bar{5}$	143	151
39	159	167	17$\bar{5}$	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
41	320	328	336	344	352	360	368	376	384	392
42	400	408	416	424	432	440	448	456	464	472
43	480	488	496	504	512	520	528	536	544	552
44	560	568	576	584	592	600	608	616	624	632
45	640	648	656	664	672	679	687	695	703	711
46	719	727	735	743	751	759	767	77$\bar{5}$	783	791
47	799	807	81$\bar{5}$	823	830	838	846	854	862	870
48	878	886	894	902	910	918	926	933	941	949
49	957	965	973	981	989	997	*00$\bar{5}$	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
51	115	123	131	139	147	155̅	162	170	178	186
52	194	202	210	218	225	233	241	249	257	265̅
53	273	280	288	296	304	312	320	327	335	343
54	351	359	367	374	382	390	398	406	414	421
55	429	437	445̅	453	461	468	476	484	492	5̅00
56	507	515	523	531	539	547	554	562	570	578
57	586	593	601	609	617	624	632	640	648	656
58	663	671	679	687	695̅	702	710	718	726	733
59	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	85̅0	858	865	873	881	889
61	896	904	912	920	927	935̅	943	950	958	966
62	974	981	989	997	*005̅	*012	*020	*028	*035	*043
63	75 051	059	066	074	082	089	097	105̅	113	120
64	128	136	143	151	159	166	174	182	189	197
65	205̅	213	220	228	236	243	251	259	266	274
66	282	289	297	305̅	312	320	328	335	343	351
67	358	366	374	381	389	397	404	412	420	427
68	435̅	442	450	458	465	473	481	488	496	504
69	511	519	526	534	542	549	557	565̅	572	580
570	587	595	603	610	618	626	633	641	648	656
71	664	671	679	686	694	702	709	717	724	732
72	740	747	755̅	762	770	778	785	793	800	808
73	815	823	831	838	846	853	861	868	876	884
74	891	899	906	914	921	929	937	944	952	959
75	967	974	982	989	997	*005̅	*012	*020	*027	*035̅
76	76 042	05̅0	057	065̅	072	080	087	095̅	103	110
77	118	125	133	140	148	155	163	170	178	185
78	193	200	208	215	223	230	238	245	253	260
79	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
81	418	425	433	440	448	455̅	462	470	477	485̅
82	492	5̅00	507	515̅	522	530	537	545̅	552	559
83	567	574	582	589	597	604	612	619	626	634
84	641	649	656	664	671	678	686	693	701	708
85	716	723	730	738	745	753	760	768	775̅	782
86	790	797	805̅	812	819	827	834	842	849	856
87	864	871	879	886	893	901	908	916	923	930
88	938	945	953	960	967	975̅	982	989	997	*004
89	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115̅	122	129	137	144	151
91	159	166	173	181	188	195	203	210	217	225̅
92	232	240	247	254	262	269	276	283	291	298
93	305	313	320	327	335̅	342	349	357	364	371
94	379	386	393	401	408	415	422	430	437	444
95	452	459	466	474	481	488	495	503	510	517
96	525̅	532	539	546	554	561	568	576	583	590
97	597	605̅	612	619	627	634	641	648	656	663
98	670	677	685̅	692	699	706	714	721	728	735
99	743	75̅0	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
600	77 815	822	830	837	844	851	859	866	873	880	
01	887	895̅	902	909	916	924	931	938	945	952	
02	960	967	974	981	988	996	*003	*010	*017	*025̅	
03	78 032	039	046	053	061	068	075̅	082	089	097	
04	104	111	118	125	132	140	147	154	161	168	
05	176	183	190	197	204	211	219	226	233	240	
06	247	254	262	269	276	283	290	297	305̅	312	8
07	319	326	333	340	347	355̅	362	369	376	383	1 0.8
08	390	398	405̅	412	419	426	433	440	447	455̅	2 1.6
09	462	469	476	483	490	497	504	512	519	526	3 2.4
610	533	540	547	554	561	569	576	583	590	597	4 3.2
11	604	611	618	625	633	640	647	654	661	668	5 4.0
12	675	682	689	696	704	711	718	725̅	732	739	6 4.8
13	746	753	760	767	774	781	789	796	803	810	7 5.6
14	817	824	831	838	845	852	859	866	873	880	8 6.4
15	888	895̅	902	909	916	923	930	937	944	951	9 7.2
16	958	965	972	979	986	993	*000	*007	*014	*021	
17	79 029	036	043	05̅0	057	064	071	078	085̅	092	
18	099	106	113	120	127	134	141	148	155	162	
19	169	176	183	190	197	204	211	218	225	232	
620	239	246	253	260	267	274	281	288	295	302	
21	309	316	323	330	337	344	351	358	365	372	7
22	379	386	393	400	407	414	421	428	435̅	442	1 0.7
23	449	456	463	470	477	484	491	498	505̅	511	2 1.4
24	518	525	532	539	546	553	560	567	574	581	3 2.1
25	588	595̅	602	609	616	623	630	637	644	650	4 2.8
26	657	664	671	678	685	692	699	706	713	720	5 3.5
27	727	734	741	748	754	761	768	775	782	789	6 4.2
28	796	803	810	817	824	831	837	844	851	858	7 4.9
29	865	872	879	886	893	900	906	913	920	927	8 5.6
630	934	941	948	955̅	962	969	975	982	989	996	9 6.3
31	80 003	010	017	024	030	037	044	051	058	065̅	
32	072	079	085	092	099	106	113	120	127	134	
33	140	147	154	161	168	175̅	182	188	195	202	
34	209	216	223	229	236	243	250	257	264	271	
35	277	284	291	298	305̅	312	318	325	332	339	
36	346	353	359	366	373	380	387	393	400	407	6
37	414	421	428	434	441	448	455̅	462	468	475	1 0.6
38	482	489	496	502	509	516	523	530	536	543	2 1.2
39	550	557	564	570	577	584	591	598	604	611	3 1.8
640	618	625̅	632	638	645	652	659	665	672	679	4 2.4
41	686	693	699	706	713	720	726	733	740	747	5 3.0
42	754	760	767	774	781	787	794	801	808	814	6 3.6
43	821	828	835̅	841	848	855̅	862	868	875	882	7 4.2
44	889	895	902	909	916	922	929	936	943	949	8 4.8
45	956	963	969	976	983	990	996	*003	*010	*017	9 5.4
46	81 023	030	037	043	050	057	064	070	077	084	
47	090	097	104	111	117	124	131	137	144	151	
48	158	164	171	178	184	191	198	204	211	218	
49	224	231	238	245̅	251	258	265̅	271	278	285̅	
650	291	298	305̅	311	318	325̅	331	338	345̅	351	
N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
650	81 291	298	305̄	311	318	325̄	331	338	345̄	351
51	358	365̄	371	378	385̄	391	398	405̄	411	418
52	425̄	431	438	445̄	451	458	465̄	471	478	485̄
53	491	498	505̄	511	518	525̄	531	538	544	551
54	558	564	571	578	584	591	598	604	611	617
55	624	631	637	644	651	657	664	671	677	684
56	690	697	704	710	717	723	730	737	743	75̄0
57	757	763	770	776	783	790	796	803	809	816
58	823	829	836	842	849	856	862	869	875	882
59	889	895	902	908	915̄	921	928	935̄	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
61	82 020	027	033	040	046	053	060	066	073	079
62	086	092	099	105	112	119	125	132	138	145̄
63	151	158	164	171	178	184	191	197	204	210
64	217	223	230	236	243	249	256	263	269	276
65	282	289	295	302	308	315̄	321	328	334	341
66	347	354	360	367	373	380	387	393	400	406
67	413	419	426	432	439	445	452	458	465̄	471
68	478	484	491	497	504	510	517	523	530	536
69	543	549	556	562	569	575	582	588	595̄	601
670	607	614	620	627	633	640	646	653	659	666
71	672	679	685	692	698	705̄	711	718	724	730
72	737	743	75̄0	756	763	769	776	782	789	795
73	802	808	814	821	827	834	840	847	853	860
74	866	872	879	885	892	898	905̄	911	918	924
75	930	937	943	95̄0	956	963	969	975	982	988
76	995̄	*001	*008	*014	*020	*027	*033	*040	*046	*052
77	83 059	065	072	078	085̄	091	097	104	110	117
78	123	129	136	142	149	155̄	161	168	174	181
79	187	193	200	206	213	219	225	232	238	245̄
680	251	257	264	270	276	283	289	296	302	308
81	315̄	321	327	334	340	347	353	359	366	372
82	378	385̄	391	398	404	410	417	423	429	436
83	442	448	455̄	461	467	474	480	487	493	499
84	506	512	518	525̄	531	537	544	550	556	563
85	569	575	582	588	594	601	607	613	620	626
86	632	639	645	651	658	664	670	677	683	689
87	696	702	708	715̄	721	727	734	740	746	753
88	759	765	771	778	784	790	797	803	809	816
89	822	828	835̄	841	847	853	860	866	872	879
690	885̄	891	897	904	910	916	923	929	935	942
91	948	954	960	967	973	979	985	992	998	*004
92	84 011	017	023	029	036	042	048	055̄	061	067
93	073	080	086	092	098	105̄	111	117	123	130
94	136	142	148	155̄	161	167	173	180	186	192
95	198	205̄	211	217	223	230	236	242	248	255̄
96	261	267	273	280	286	292	298	305̄	311	317
97	323	330	336	342	348	354	361	367	373	379
98	386	392	398	404	410	417	423	429	435	442
99	448	454	460	466	473	479	485̄	491	497	504
700	510	516	522	528	535̄	541	547	553	559	566
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535̅	541	547	553	559	566
01	572	578	584	590	597	603	609	615	621	628
02	634	640	646	652	658	665̅	671	677	683	689
03	696	702	708	714	720	726	733	739	745̅	751
04	757	763	770	776	782	788	794	800	807	813
05	819	825	831	837	844	85̅0	856	862	868	874
06	880	887	893	899	905	911	917	924	930	936
07	942	948	954	960	967	973	979	985̅	991	997
08	85 003	009	016	022	028	034	040	046	052	058
09	065̅	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175̅	181
11	187	193	199	205	211	217	224	230	236	242
12	248	254	260	266	272	278	285̅	291	297	303
13	309	315	321	327	333	339	345	352	358	364
14	370	376	382	388	394	400	406	412	418	425̅
15	431	437	443	449	455̅	461	467	473	479	485
16	491	497	503	509	516	522	528	534	540	546
17	552	558	564	570	576	582	588	594	600	606
18	612	618	625̅	631	637	643	649	655̅	661	667
19	673	679	685̅	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
21	794	800	806	812	818	824	830	836	842	848
22	854	860	866	872	878	884	890	896	902	908
23	914	920	926	932	938	944	95̅0	956	962	968
24	974	980	986	992	998	*004	*010	*016	*022	*028
25	86 034	040	046	052	058	064	070	076	082	088
26	094	100	106	112	118	124	130	136	141	147
27	153	159	165	171	177	183	189	195	201	207
28	213	219	225	231	237	243	249	255̅	261	267
29	273	279	285̅	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
31	392	390	404	410	415	421	427	433	439	445
32	451	457	463	469	475̅	481	487	493	499	504
33	510	516	522	528	534	540	546	552	558	564
34	570	576	581	587	593	599	605	611	617	623
35	629	635̅	641	646	652	658	664	670	676	682
36	688	694	700	705	711	717	723	729	735̅	741
37	747	753	759	764	770	776	782	788	794	800
38	806	812	817	823	829	835	841	847	853	859
39	864	870	876	882	888	894	900	906	911	917
740	923	929	935̅	941	947	953	958	964	970	976
41	982	988	994	999	*005	*011	*017	*023	*029	*035̅
42	87 040	046	052	058	064	070	075	081	087	093
43	099	105̅	111	116	122	128	134	140	146	151
44	157	163	169	175̅	181	186	192	198	204	210
45	216	221	227	233	239	245̅	251	256	262	268
46	274	280	286	291	297	303	309	315̅	320	326
47	332	338	344	349	355	361	367	373	379	384
48	390	396	402	408	413	419	425̅	431	437	442
49	448	454	460	466	471	477	483	489	495̅	500
750	506	512	518	523	529	535	541	547	552	558
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
750	87 506	512	518	523	529	535	541	547	552	558	
51	564	570	576	581	587	593	599	604	610	616	
52	622	628	633	639	64$\bar{5}$	651	656	662	668	674	
53	679	685	691	697	703	708	714	720	726	731	
54	737	743	749	754	760	766	772	777	783	789	
55	79$\bar{5}$	800	806	812	818	823	829	83$\bar{5}$	841	846	
56	852	858	864	869	875	881	887	892	898	904	
57	910	915	921	927	933	938	944	9$\bar{5}$0	955	961	
58	967	973	978	984	990	996	*001	*007	*013	*018	
59	88 024	030	036	041	047	053	058	064	070	076	
760	081	087	093	098	104	110	116	121	127	133	
61	138	144	1$\bar{5}$0	156	161	167	173	178	184	190	**6**
62	195	201	207	213	218	224	230	235	241	247	1 0.6
63	252	258	264	270	275	281	287	292	298	304	2 1.2
64	309	315	321	326	332	338	343	349	35$\bar{5}$	360	3 1.8
65	366	372	377	383	389	39$\bar{5}$	400	406	412	417	4 2.4
66	423	429	434	440	446	451	457	463	468	474	5 3.0
67	480	485	491	497	502	508	513	519	52$\bar{5}$	530	6 3.6
68	536	542	547	553	559	564	570	576	581	587	7 4.2
69	593	598	604	610	615	621	627	632	638	643	8 4.8
770	649	65$\bar{5}$	660	666	672	677	683	689	694	700	9 5.4
71	705	711	717	722	728	734	739	74$\bar{5}$	750	756	
72	762	767	773	779	784	790	795	801	807	812	
73	818	824	829	83$\bar{5}$	840	846	852	857	863	868	
74	874	880	885	891	897	902	908	913	919	92$\bar{5}$	
75	930	936	941	947	953	958	964	969	97$\bar{5}$	981	
76	986	992	997	*003	*009	*014	*020	*025	*031	*037	
77	89 042	048	053	059	064	070	076	081	087	092	
78	098	104	109	11$\bar{5}$	120	126	131	137	143	148	
79	154	159	16$\bar{5}$	170	176	182	187	193	198	204	
780	209	215	221	226	232	237	243	248	254	260	
81	265	271	276	282	287	293	298	304	310	315	**5**
82	321	326	332	337	343	348	354	360	365	371	1 0.5
83	376	382	387	393	398	404	409	41$\bar{5}$	421	426	2 1.0
84	432	437	443	448	454	459	46$\bar{5}$	470	476	481	3 1.5
85	487	492	498	504	509	51$\bar{5}$	520	526	531	537	4 2.0
86	542	548	553	559	564	570	575	581	586	592	5 2.5
87	597	603	609	614	620	625	631	636	642	647	6 3.0
88	653	658	664	669	67$\bar{5}$	680	686	691	697	702	7 3.5
89	708	713	719	724	730	735	741	746	752	757	8 4.0
790	763	768	774	779	78$\bar{5}$	790	796	801	807	812	9 4.5
91	818	823	829	834	840	845	851	856	862	867	
92	873	878	883	889	894	900	905	911	916	922	
93	927	933	938	944	949	95$\bar{5}$	960	966	971	977	
94	982	988	993	998	*004	*009	*01$\bar{5}$	*020	*026	*031	
95	90 037	042	048	053	059	064	069	07$\bar{5}$	080	086	
96	091	097	102	108	113	119	124	129	13$\bar{5}$	140	
97	146	151	157	162	168	173	179	184	189	19$\bar{5}$	
98	200	206	211	217	222	227	233	238	244	249	
99	25$\bar{5}$	260	266	271	276	282	287	293	298	304	
800	309	314	320	325	331	336	342	347	352	358	
N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
800	90 309	314	320	325	331	336	342	347	352	358	
01	363	369	374	380	385̄	390	396	401	407	412	
02	417	423	428	434	439	445̄	45̄0	455	461	466	
03	472	477	482	488	493	499	504	509	515̄	520	
04	526	531	536	542	547	553	558	563	569	574	
05	580	585̄	590	596	601	607	612	617	623	628	
06	634	639	644	65̄0	655	660	666	671	677	682	
07	687	693	698	703	709	714	720	725	730	736	
08	741	747	752	757	763	768	773	779	784	789	
09	795̄	800	806	811	816	822	827	832	838	843	
810	849	854	859	865̄	870	875	881	886	891	897	
11	902	907	913	918	924	929	934	940	945̄	950	6
12	956	961	966	972	977	982	988	993	998	*004	1 0.6
13	91 009	014	020	025	030	036	041	046	052	057	2 1.2
14	062	068	073	078	084	089	094	100	105	110	3 1.8
15	116	121	126	132	137	142	148	153	158	164	4 2.4
16	169	174	180	185̄	190	196	201	206	212	217	5 3.0
17	222	228	233	238	243	249	254	259	265̄	270	6 3.6
18	275	281	286	291	297	302	307	312	318	323	7 4.2
19	328	334	339	344	35̄0	355̄	360	365	371	376	8 4.8
820	381	387	392	397	403	408	413	418	424	429	9 5.4
21	434	440	445̄	450	455	461	466	471	477	482	
22	487	492	498	503	508	514	519	524	529	535̄	
23	540	545	551	556	561	566	572	577	582	587	
24	593	598	603	609	614	619	624	630	635̄	640	
25	645	651	656	661	666	672	677	682	687	693	
26	698	703	709	714	719	724	730	735̄	740	745	
27	751	756	761	766	772	777	782	787	793	798	
28	803	808	814	819	824	829	834	840	845̄	850	
29	855	861	866	871	876	882	887	892	897	903	
830	908	913	918	924	929	934	939	944	95̄0	955̄	
31	960	965	971	976	981	986	991	997	*002	*007	5
32	92 012	018	023	028	033	038	044	049	054	059	1 0.5
33	065̄	070	075̄	080	085	091	096	101	106	111	2 1.0
34	117	122	127	132	137	143	148	153	158	163	3 1.5
35	169	174	179	184	189	195̄	200	205	210	215	4 2.0
36	221	226	231	236	241	247	252	257	262	267	5 2.5
37	273	278	283	288	293	298	304	309	314	319	6 3.0
38	324	330	335̄	340	345	350	355	361	366	371	7 3.5
39	376	381	387	392	397	402	407	412	418	423	8 4.0
840	428	433	438	443	449	454	459	464	469	474	9 4.5
41	480	485̄	490	495	500	505	511	516	521	526	
42	531	536	542	547	552	557	562	567	572	578	
43	583	588	593	598	603	609	614	619	624	629	
44	634	639	645̄	65̄0	655̄	660	665	670	675	681	
45	686	691	696	701	706	711	716	722	727	732	
46	737	742	747	752	758	763	768	773	778	783	
47	788	793	799	804	809	814	819	824	829	834	
48	840	845̄	85̄0	855̄	860	865	870	875	881	886	
49	891	896	901	906	911	916	921	927	932	937	
850	942	947	952	957	962	967	973	978	983	988	
N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
850	92 942	947	952	957	962	967	973	978	983	988	
51	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
52	93 044	049	054	059	064	069	075̄	080	085̄	090	
53	095̄	100	105	110	115	120	125	131	136	141	
54	146	151	156	161	166	171	176	181	186	192	
55	197	202	207	212	217	222	227	232	237	242	
56	247	252	258	263	268	273	278	283	288	293	
57	298	303	308	313	318	323	328	334	339	344	
58	349	354	359	364	369	374	379	384	389	394	
59	399	404	409	414	420	425̄	430	435̄	440	445̄	
860	45̄0	455̄	460	465̄	470	475	480	485	490	495	
61	500	505	510	515	520	526	531	536	541	546	
62	551	556	561	566	571	576	581	586	591	596	
63	601	606	611	616	621	626	631	636	641	646	
64	651	656	661	666	671	676	682	687	692	697	
65	702	707	712	717	722	727	732	737	742	747	
66	752	757	762	767	772	777	782	787	792	797	
67	802	807	812	817	822	827	832	837	842	847	
68	852	857	862	867	872	877	882	887	892	897	
69	902	907	912	917	922	927	932	937	942	947	
870	952	957	962	967	972	977	982	987	992	997	
71	94 002	007	012	017	022	027	032	037	042	047	
72	052	057	062	067	072	077	082	086	091	096	
73	101	106	111	116	121	126	131	136	141	146	
74	151	156	161	166	171	176	181	186	191	196	
75	201	206	211	216	221	226	231	236	240	245	
76	250	255	260	265	270	275	280	285	290	295	
77	300	305̄	310	315̄	320	325̄	330	335̄	340	345̄	
78	349	354	359	364	369	374	379	384	389	394	
79	399	404	409	414	419	424	429	433	438	443	
880	448	453	458	463	468	473	478	483	488	493	
81	498	503	507	512	517	522	527	532	537	542	
82	547	552	557	562	567	571	576	581	586	591	
83	596	601	606	611	616	621	626	630	635	640	
84	645	650	655	660	665̄	670	675̄	680	685̄	689	
85	694	699	704	709	714	719	724	729	734	738	
86	743	748	753	758	763	768	773	778	783	787	
87	792	797	802	807	812	817	822	827	832	836	
88	841	846	851	856	861	866	871	876	880	885	
89	890	895	900	905̄	910	915̄	919	924	929	934	
890	939	944	949	954	959	963	968	973	978	983	
91	988	993	998	*002	*007	*012	*017	*022	*027	*032	
92	95 036	041	046	051	056	061	066	071	075	080	
93	085	090	095̄	100	105̄	109	114	119	124	129	
94	134	139	143	148	153	158	163	168	173	177	
95	182	187	192	197	202	207	211	216	221	226	
96	231	236	240	245	250	255	260	265̄	270	274	
97	279	284	289	294	299	303	308	313	318	323	
98	328	332	337	342	347	352	357	361	366	371	
99	376	381	386	390	395	400	405̄	410	415̄	419	
900	424	429	434	439	444	448	453	458	463	468	
N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
900	95 424	429	434	439	444	448	453	458	463	468	
01	472	477	482	487	492	497	501	506	511	516	
02	521	525	530	535	540	545̄	55̄0	554	559	564	
03	569	574	578	583	588	593	598	602	607	612	
04	617	622	626	631	636	641	646	650	655	660	
05	665̄	670	674	679	684	689	694	698	703	708	
06	713	718	722	727	732	737	742	746	751	756	
07	761	766	770	775	780	785̄	789	794	799	804	
08	809	813	818	823	828	832	837	842	847	852	
09	856	861	866	871	875	880	885	890	895̄	899	
910	904	909	914	918	923	928	933	938	942	947	
11	952	957	961	966	971	976	980	985	990	995̄	5 1 0.5 2 1.0 3 1.5 4 2.0 5 2.5 6 3.0 7 3.5 8 4.0 9 4.5
12	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	
13	96 047	052	057	061	066	071	076	080	085	090	
14	095̄	099	104	109	114	118	123	128	133	137	
15	142	147	152	156	161	166	171	175	180	185̄	
16	190	194	199	204	209	213	218	223	227	232	
17	237	242	246	251	256	261	265	270	275̄	280	
18	284	289	294	298	303	308	313	317	322	327	
19	332	336	341	346	350	355	360	365̄	369	374	
920	379	384	388	393	398	402	407	412	417	421	
21	426	431	435	440	445̄	45̄0	454	459	464	468	
22	473	478	483	487	492	497	501	506	511	515	
23	520	525̄	530	534	539	544	548	553	558	562	
24	567	572	577	581	586	591	595	600	605̄	609	
25	614	619	624	628	633	638	642	647	652	656	
26	661	666	670	675	680	685̄	689	694	699	703	
27	708	713	717	722	727	731	736	741	745	750	
28	755̄	759	764	769	774	778	783	788	792	797	
29	802	806	811	816	820	825̄	830	834	839	844	
930	848	853	858	862	867	872	876	881	886	890	
31	895̄	900	904	909	914	918	923	928	932	937	4 1 0.4 2 0.8 3 1.2 4 1.6 5 2.0 6 2.4 7 2.8 8 3.2 9 3.6
32	942	946	951	956	960	965̄	970	974	979	984	
33	988	993	997	*002	*007	*011	*016	*021	*025	*030	
34	97 035̄	039	044	049	053	058	063	067	072	077	
35	081	086	090	095	100	104	109	114	118	123	
36	128	132	137	142	146	151	155	160	165̄	169	
37	174	179	183	188	192	197	202	206	211	216	
38	220	225̄	230	234	239	243	248	253	257	262	
39	267	271	276	280	285	290	294	299	304	308	
940	313	317	322	327	331	336	340	345	35̄0	354	
41	359	364	368	373	377	382	387	391	396	400	
42	405	410	414	419	424	428	433	437	442	447	
43	451	456	460	465̄	470	474	479	483	488	493	
44	497	502	506	511	516	520	525̄	529	534	539	
45	543	548	552	557	562	566	571	575	580	585̄	
46	589	594	598	603	607	612	617	621	626	630	
47	635̄	640	644	649	653	658	663	667	672	676	
48	681	685	690	695̄	699	704	708	713	717	722	
49	727	731	736	740	745̄	749	754	759	763	768	
950	772	777	782	786	791	795	800	804	809	813	
N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
51	818	823	827	832	836	841	845	850	855̄	859
52	864	868	873	877	882	886	891	896	900	905̄
53	909	914	918	923	928	932	937	941	946	950
54	955̄	959	964	968	973	978	982	987	991	996
55	98 000	005̄	009	014	019	023	028	032	037	041
56	046	050	055̄	059	064	068	073	078	082	087
57	091	096	100	105̄	109	114	118	123	127	132
58	137	141	146	150	155̄	159	164	168	173	177
59	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	25̄0	254	259	263	268
61	272	277	281	286	290	295̄	299	304	308	313
62	318	322	327	331	336	340	345̄	349	354	358
63	363	367	372	376	381	385	390	394	399	403
64	408	412	417	421	426	430	435̄	439	444	448
65	453	457	462	466	471	475	480	484	489	493
66	498	502	507	511	516	520	525̄	529	534	538
67	543	547	552	556	561	565	570	574	579	583
68	588	592	597	601	605	610	614	619	623	628
69	632	637	641	646	650	655̄	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
71	722	726	731	735	740	744	749	753	758	762
72	767	771	776	780	784	789	793	798	802	807
73	811	816	820	825̄	829	834	838	843	847	851
74	856	860	865̄	869	874	878	883	887	892	896
75	900	905̄	909	914	918	923	927	932	936	941
76	945̄	949	954	958	963	967	972	976	981	985
77	989	994	998	*003	*007	*012	*016	*021	*025	*029
78	99 034	038	043	047	052	056	061	065̄	069	074
79	078	083	087	092	096	100	105̄	109	114	118
980	123	127	131	136	140	145̄	149	154	158	162
81	167	171	176	180	185̄	189	193	198	202	207
82	211	216	220	224	229	233	238	242	247	251
83	255	260	264	269	273	277	282	286	291	295
84	300	304	308	313	317	322	326	330	335̄	339
85	344	348	352	357	361	366	370	374	379	383
86	388	392	396	401	405	410	414	419	423	427
87	432	436	441	445̄	449	454	458	463	467	471
88	476	480	484	489	493	498	502	506	511	515
89	520	524	528	533	537	542	546	550	555̄	559
990	564	568	572	577	581	585	590	594	599	603
91	607	612	616	621	625̄	629	634	638	642	647
92	651	656	660	664	669	673	677	682	686	691
93	695̄	699	704	708	712	717	721	726	730	734
94	739	743	747	752	756	760	765̄	769	774	778
95	782	787	791	795	800	804	808	813	817	822
96	826	830	835̄	839	843	848	852	856	861	865
97	870	874	878	883	887	891	896	900	904	909
98	913	917	922	926	930	935̄	939	944	948	952
99	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035̄	039
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
1000	000 000	043	087	130	174	217	260	304	347	391
1001	434	477	521	564	608	651	694	738	781	824
1002	868	911	954	998	*041	*084	*128	*171	*214	*258
1003	001 301	344	388	431	474	517	561	604	647	690
1004	734	777	820	863	907	950	993	*036	*080	*123
1005	002 166	209	252	296	339	382	425	468	512	555
1006	598	641	684	727	771	814	857	900	943	986
1007	003 029	073	116	159	202	245	288	331	374	417
1008	461	504	547	590	633	676	719	762	805	848
1009	891	934	977	*020	*063	*106	*149	*192	*235	*278
1010	004 321	364	407	450	493	536	579	622	665	708
1011	751	794	837	880	923	966	*009	*052	*095	*138
1012	005 180	223	266	309	352	395	438	481	524	567
1013	609	652	695	738	781	824	867	909	952	995
1014	006 038	081	124	166	209	252	295	338	380	423
1015	466	509	552	594	637	680	723	765	808	851
1016	894	936	979	*022	*065	*107	*150	*193	*236	*278
1017	007 321	364	406	449	492	534	577	620	662	705
1018	748	790	833	876	918	961	*004	*046	*089	*132
1019	008 174	217	259	302	345	387	430	472	515	558
1020	600	643	685	728	770	813	856	898	941	983
1021	009 026	068	111	153	196	238	281	323	366	408
1022	451	493	536	578	621	663	706	748	791	833
1023	876	918	961	*003	*045	*088	*130	*173	*215	*258
1024	010 300	342	385	427	470	512	554	597	639	681
1025	724	766	809	851	893	936	978	*020	*063	*105
1026	011 147	190	232	274	317	359	401	444	486	528
1027	570	613	655	697	740	782	824	866	909	951
1028	993	*035	*078	*120	*162	*204	*247	*289	*331	*373
1029	012 415	458	500	542	584	626	669	711	753	795
1030	837	879	922	964	*006	*048	*090	*132	*174	*217
1031	013 259	301	343	385	427	469	511	553	596	638
1032	680	722	764	806	848	890	932	974	*016	*058
1033	014 100	142	184	226	268	310	352	395	437	479
1034	521	563	605	647	689	730	772	814	856	898
1035	940	982	*024	*066	*108	*150	*192	*234	*276	*318
1036	015 360	402	444	485	527	569	611	653	$69\bar{5}$	737
1037	779	821	863	904	946	988	*030	*072	*114	*156
1038	016 197	239	281	323	365	407	448	490	532	574
1039	616	657	699	741	783	824	866	908	950	992
1040	017 033	075	117	159	200	242	284	326	367	409
1041	451	492	534	576	618	659	701	743	784	826
1042	868	909	951	993	*034	*076	*118	*159	*201	*243
1043	018 284	326	368	409	451	492	534	576	617	659
1044	700	742	784	825	867	908	950	992	*033	*075
1045	019 116	158	199	241	282	324	366	407	449	490
1046	532	573	615	656	698	739	781	822	864	905
1047	947	988	*030	*071	*113	*154	*195	*237	*278	*320
1048	020 361	403	444	486	527	568	610	651	693	734
1049	775	817	858	900	941	982	*024	*065	*107	*148
1050	021 189	231	272	313	355	396	437	479	520	561
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.	44	43	42	41
1	4.4	4.3	4.2	4.1
2	8.8	8.6	8.4	8.2
3	13.2	12.9	12.6	12.3
4	17.6	17.2	16.8	16.4
5	22.0	21.5	21.0	20.5
6	26.4	25.8	25.2	24.6
7	30.8	30.1	29.4	28.7
8	35.2	34.4	33.6	32.8
9	39.6	38.7	37.8	36.9

Prop. Pts.

Google

N.	0	1	2	3	4	5	6	7	8	9
1050	021 189	231	272	313	355	396	437	479	520	561
1051	603	644	685	727	768	809	851	892	933	974
1052	022 016	057	098	140	181	222	263	305	346	387
1053	428	470	511	552	593	635	676	717	758	799
1054	841	882	923	964	*005	*047	*088	*129	*170	*211
1055	023 252	294	335	376	417	458	499	541	582	623
1056	664	705	746	787	828	870	911	952	993	*034
1057	024 075̄	116	157	198	239	280	321	363	404	445
1058	486	527	568	609	650	691	732	773	814	855
1059	896	937	978	*019	*060	*101	*142	*183	*224	*265
1060	025 306	347	388	429	470	511	552	593	634	674
1061	715	756	797	838	879	920	961	*002	*043	*084
1062	26 125	165	206	247	288	329	370	411	452	492
1063	533	574	615̄	656	697	737	778	819	860	901
1064	942	982	*023	*064	*105	*146	*186	*227	*268	*309
1065	027 350	390	431	472	513	553	594	635̄	676	716
1066	757	798	839	879	920	961	*002	*042	*083	*124
1067	028 164	205	246	287	327	368	409	449	490	531
1068	571	612	653	693	734	775	815	856	896	937
1069	978	*018	*059	*100	*140	*181	*221	*262	*303	*343
1070	029 384	424	465	506	546	587	627	668	708	749
1071	789	830	871	911	952	992	*033	*073	*114	*154
1072	030 195	235	276	316	357	397	438	478	519	559
1073	600	640	681	721	762	802	843	883	923	964
1074	031 004	045	085	126	166	206	247	287	328	368
1075	408	449	489	530	570	610	651	691	732	772
1076	812	853	893	933	974	*014	*054	*095	*135	*175
1077	032 216	256	296	337	377	417	458	498	538	578
1078	619	659	699	740	780	820	860	901	941	981
1079	033 021	062	102	142	182	223	263	303	343	384
1080	424	464	504	544	585	625	665̄	705	745	785
1081	826	866	906	946	986	*027	*067	*107	*147	*187
1082	034 227	267	308	348	388	428	468	508	548	588
1083	628	669	709	749	789	829	869	909	949	989
1084	035 029	069	109	149	190	230	270	310	350	390
1085	430	470	510	550	590	630	670	710	750	790
1086	830	870	910	950	990	*030	*070	*110	*150	*190
1087	036 230	269	309	349	389	429	469	509	549	589
1088	629	669	709	749	789	828	868	908	948	988
1089	037 028	068	108	148	187	227	267	307	347	387
1090	426	466	506	546	586	626	665	705	745	785
1091	825	865	904	944	984	*024	*064	*103	*143	*183
1092	038 223	262	302	342	382	421	461	501	541	580
1093	620	660	700	739	779	819	859	898	938	978
1094	039 017	057	097	136	176	216	255	295	335	374
1095	414	454	493	533	573	612	652	692	731	771
1096	811	850	890	929	969	*009	*048	*088	*127	*167
1097	040 207	246	286	325	365̄	405	444	484	523	563
1098	602	642	681	721	761	800	840	879	919	958
1099	998	*037	*077	*116	*156	*195	*235	*274	*314	*353
1100	041 393	432	472	511	551	590	630	669	708	748
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	42
1	4.2
2	8.4
3	12.6
4	16.8
5	21.0
6	25.2
7	29.4
8	33.6
9	37.8

	41
1	4.1
2	8.2
3	12.3
4	16.4
5	20.5
6	24.6
7	28.7
8	32.8
9	36.9

	40
1	4.0
2	8.0
3	12.0
4	16.0
5	20.0
6	24.0
7	28.0
8	32.0
9	36.0

	39
1	3.9
2	7.8
3	11.7
4	15.6
5	19.5
6	23.4
7	27.3
8	31.2
9	35.1

Prop. Pts.

TABLE II.

CONSTANTS WITH THEIR LOGARITHMS.

	Number.	Logarithm.
Ratio of circumference to diameter, π,	3.14159265	0.49714 99
.. .. .. π^2,	9.86960440	0.99429 97
.. .. .. 2π,	6.28318531	0.79817 99
.. .. .. $\sqrt{\pi}$,	1.77245385	0.24857 49
Number of degrees in circumference,	360°	2.55630 25
.. minutes ..	21600′	4.33445 38
.. seconds ..	1296000″	6.11260 50
Degrees in arc equal to radius,	57°.2957795	1.75812 26
Minutes	3437′.74677	3.53627 39
Seconds	206264″.806	5.31442 51
Length of arc of 1 degree,	.01745329	8.24187 74—10
.. .. 1 minute,	.00029089	6.46372 61—10
.. .. 1 second,	.000004848	4.68557 49—10
Number of hours in 1 day,	24	1.38021 12
.. minutes ..	1440	3.15836 25
.. seconds .	86400	4.93651 37
Number of days in Julian year,	365.25	2.56259 02
Naperian base,	2.718281828	0.43429 45
Modulus of common logarithms,	0.434294482	9.63778 43—10
Hours in which earth revolves through arc equal to radius,	3.8197186	0.58203 14
Minutes of time	229.18312	2.36018 26
Seconds of time	13750.987	4.13833 39

TABLE III.

FOR

SINES AND TANGENTS OF SMALL ANGLES.

TO FIND THE SINE OR TANGENT:

$\text{Log sin } \alpha = \log \alpha \text{ (in seconds)} + S.$

$\text{Log tan } \alpha = \log \alpha \text{ (in seconds)} + T.$

TO FIND A SMALL ANGLE FROM ITS SINE OR TANGENT:

$\text{Log } \alpha \text{ (in seconds)} = \log \sin \alpha + S'.$

$\text{Log } \alpha \text{ (in seconds)} = \log \tan \alpha + T'.$

0°

″	′	L. Sin.	S	T	S′	T′
0	0	—	4.68557	4.68557	5.31443	5.31443
60	1	6.46373	.68557	.68557	.31443	.31443
120	2	.76476	.68557	.68557	.31443	.31443
180	3	.94085	.68557	.68557	.31443	.31443
240	4	7.06579	.68557	.68558	.31443	.31442
300	5	7.16270	4.68557	4.68558	5.31443	5.31442
360	6	.24188	.68557	.68558	31443	.31442
420	7	.30882	.68557	.68558	.31443	.31442
480	8	.36682	.68557	.68558	.31443	.31442
540	9	.41797	.68557	.68558	.31443	.31442
600	10	7.46373	4.68557	4.68558	5.31443	5.31442
660	11	.50512	.68557	.68558	.31443	.31442
720	12	.54291	.68557	.68558	.31443	.31442
780	13	.57767	.68557	.68558	.31443	.31442
840	14	.60985	.68557	.68558	.31443	.31442
900	15	7.63982	4.68557	4.68558	5.31443	5.31442
960	16	.66784	.68557	.68558	.31443	.31442
1020	17	.69417	.68557	.68558	.31443	.31442
1080	18	.71900	.68557	.68558	.31443	.31442
1140	19	.74248	.68557	.68558	.31443	.31442
1200	20	7.76475	4.68557	4.68558	5.31443	5.31442
1260	21	.78594	.68557	.68558	.31443	.31442
1320	22	.80615	.68557	.68558	.31443	.31442
1380	23	.82545	.68557	.68558	.31443	.31442
1440	24	.84393	.68557	.68558	.31443	.31442
1500	25	7.86166	4.68557	4.68558	5.31443	5.31442
1560	26	.87870	.68557	.68558	.31443	.31442
1620	27	.89509	.68557	.68558	.31443	.31442
1680	28	.91088	.68557	.68558	.31443	.31442
1740	29	.92612	.68557	.68559	.31443	.31441
1800	30	7.94084	4.68557	4.68559	5.31443	5.31441
1860	31	.95508	.68557	.68559	.31443	.31441
1920	32	.96887	.68557	.68559	.31443	.31441
1980	33	.98223	.68557	.68559	.31443	.31441
2040	34	.99520	.68557	.68559	.31443	.31441
2100	35	8.00779	4.68557	4.68559	5.31443	5.31441
2160	36	.02002	.68557	.68559	.31443	.31441
2220	37	.03192	.68557	.68559	.31443	.31441
2280	38	.04350	.68557	.68559	.31443	.31441
2340	39	.05478	.68557	.68559	.31443	.31441
2400	40	8.06578	4.68557	4.68559	5.31443	5.31441
2460	41	.07650	.68556	.68560	.31444	.31440
2520	42	.08696	.68556	.68560	.31444	.31440
2580	43	.09718	.68556	.68560	.31444	.31440
2640	44	.10717	.68556	.68560	.31444	.31440
2700	45	8.11693	4.68556	4.68560	5.31444	5.31440
2760	46	.12647	.68556	.68560	.31444	.31440
2820	47	.13581	.68556	.68560	.31444	.31440
2880	48	.14495	.68556	.68560	.31444	.31440
2940	49	.15391	.68556	.68560	.31444	.31440
3000	50	8.16268	4.68556	4.68561	5.31444	5.31439
3060	51	.17128	.68556	.68561	.31444	.31439
3120	52	.17971	.68556	.68561	.31444	.31439
3180	53	.18798	.68556	.68561	.31444	.31439
3240	54	.19610	.68556	.68561	.31444	.31439
3300	55	8.20407	4.68556	4.68561	5.31444	5.31439
3360	56	.21189	.68556	.68561	.31444	.31439
3420	57	.21958	.68555	.68561	.31445	.31439
3480	58	.22713	.68555	.68562	.31445	.31438
3540	59	.23456	.68555	.68562	.31445	.31438
3600	60	8.24186	4.68555	4.68562	5.31445	5.31438

1°

″	′	L. Sin.	S	T	S′	T′
3600	0	8.24186	4.68555	4.68562	5.31445	5.31438
3660	1	.24903	.68555	.68562	.31445	.31438
3720	2	.25609	.68555	.68562	.31445	.31438
3780	3	.26304	.68555	.68562	.31445	.31438
3840	4	.26988	.68555	.68563	.31445	.31437
3900	5	8.27661	4.68555	4.68563	5.31445	5.31437
3960	6	.28324	.68555	.68563	.31445	.31437
4020	7	.28977	.68555	.68563	.31445	.31437
4080	8	.29621	.68555	.68563	.31445	.31437
4140	9	.30255	.68555	.68563	.31445	.31437
4200	10	8.30879	4.68554	4.68563	5.31446	5.31437
4260	11	.31495	.68554	.68564	.31446	.31436
4320	12	.32103	.68554	.68564	.31446	.31436
4380	13	.32702	.68554	.68564	.31446	.31436
4440	14	.33292	.68554	.68564	.31446	.31436
4500	15	8.33875	4.68554	4.68564	5.31446	5.31436
4560	16	.34450	.68554	.68565	.31446	.31435
4620	17	.35018	.68554	.68565	.31446	.31435
4680	18	.35578	.68554	.68565	.31446	.31435
4740	19	.36131	.68554	.68565	.31446	.31435
4800	20	8.36678	4.68554	4.68565	5.31446	5.31435
4860	21	.37217	.68553	.68566	.31447	.31434
4920	22	.37750	.68553	.68566	.31447	.31434
4980	23	.38276	.68553	.68566	.31447	.31434
5040	24	.38796	.68553	.68566	.31447	.31434
5100	25	8.39310	4.68553	4.68566	5.31447	5.31434
5160	26	.39818	.68553	.68567	.31447	.31433
5220	27	.40320	.68553	.68567	.31447	.31433
5280	28	.40816	.68553	.68567	.31447	.31433
5340	29	.41307	.68553	.68567	.31447	.31433
5400	30	8.41792	4.68553	4.68567	5.31447	5.31433
5460	31	.42272	.68552	.68568	.31448	.31432
5520	32	.42746	.68552	.68568	.31448	.31432
5580	33	.43216	.68552	.68568	.31448	.31432
5640	34	.43680	.68552	.68568	.31448	.31432
5700	35	8.44139	4.68552	4.68569	5.31448	5.31431
5760	36	.44594	.68552	.68569	.31448	.31431
5820	37	.45044	.68552	.68569	.31448	.31431
5880	38	.45489	.68552	.68569	.31448	.31431
5940	39	.45930	.68551	.68569	.31449	.31431
6000	40	8.46366	4.68551	4.68570	5.31449	5.31430
6060	41	.46799	.68551	.68570	.31449	.31430
6120	42	.47226	.68551	.68570	.31449	.31430
6180	43	.47650	.68551	.68570	.31449	.31430
6240	44	.48069	.68551	.68571	.31449	.31429
6300	45	8.48485	4.68551	4.68571	5.31449	5.31429
6360	46	.48896	.68551	.68571	.31449	.31429
6420	47	.49304	.68550	.68572	.31450	.31428
6480	48	.49708	.68550	.68572	.31450	.31428
6540	49	.50108	.68550	.68572	.31450	.31428
6600	50	8.50504	4.68550	4.68572	5.31450	5.31428
6660	51	.50897	.68550	.68573	.31450	.31427
6720	52	.51287	.68550	.68573	.31450	.31427
6780	53	.51673	.68550	.68573	.31450	.31427
6840	54	.52055	.68550	.68573	.31450	.31427
6900	55	8.52434	4.68549	4.68574	5.31451	5.31426
6960	56	.52810	.68549	.68574	.31451	.31426
7020	57	.53183	.68549	.68574	.31451	.31426
7080	58	.53552	.68549	.68575	.31451	.31425
7140	59	.53919	.68549	.68575	.31451	.31425
7200	60	8.54282	4.68549	4.68575	5.31451	5.31425

TABLE IV.

LOGARITHMS

OF THE

SINE, COSINE, TANGENT AND COTANGENT

FOR

EACH MINUTE OF THE QUADRANT.

0°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	—		—		—	0.00 000	60
1	6.46 373	30103	6.46 373	30103	3.53 627	0.00 000	59
2	6.76 476	17609	6.76 476	17609	3.23 524	0.00 000	58
3	6.94 085	12494	6.94 085	12494	3.05 915	0.00 000	57
4	7.06 579	9691	7.06 579	9691	2.93 421	0.00 000	56
5	7.16 270	7918	7.16 270	7918	2.83 730	0.00 000	55
6	7.24 188	6694	7.24 188	6694	2.75 812	0.00 000	54
7	7.30 882	5800	7.30 882	5800	2.69 118	0.00 000	53
8	7.36 682	5115	7.36 682	5115	2.63 318	0.00 000	52
9	7.41 797	4576	7.41 797	4576	2.58 203	0.00 000	51
10	7.46 373	4139	7.46 373	4139	2.53 627	0.00 000	50
11	7.50 512	3779	7.50 512	3779	2.49 488	0.00 000	49
12	7.54 291	3476	7.54 291	3476	2.45 709	0.00 000	48
13	7.57 767	3218	7.57 767	3219	2.42 233	0.00 000	47
14	7.60 985	2997	7.60 986	2996	2.39 014	0.00 000	46
15	7.63 982	2802	7.63 982	2803	2.36 018	0.00 000	45
16	7.66 784	2633	7.66 785	2633	2.33 215	0.00 000	44
17	7.69 417	2483	7.69 418	2482	2.30 582	9.99 999	43
18	7.71 900	2348	7.71 900	2348	2.28 100	9.99 999	42
19	7.74 248	2227	7.74 248	2228	2.25 752	9.99 999	41
20	7.76 475	2119	7.76 476	2119	2.23 524	9.99 999	40
21	7.78 594	2021	7.78 595	2020	2.21 405	9.99 999	39
22	7.80 615	1930	7.80 615	1931	2.19 385	9.99 999	38
23	7.82 545	1848	7.82 546	1848	2.17 454	9.99 999	37
24	7.84 393	1773	7.84 394	1773	2.15 606	9.99 999	36
25	7.86 166	1704	7.86 167	1704	2.13 833	9.99 999	35
26	7.87 870	1639	7.87 871	1639	2.12 129	9.99 999	34
27	7.89 509	1579	7.89 510	1579	2.10 490	9.99 999	33
28	7.91 088	1524	7.91 089	1524	2.08 911	9.99 999	32
29	7.92 612	1472	7.92 613	1473	2.07 387	9.99 998	31
30	7.94 084	1424	7.94 086	1424	2.05 914	9.99 998	30
31	7.95 508	1379	7.95 510	1379	2.04 490	9.99 998	29
32	7.96 887	1336	7.96 889	1336	2.03 111	9.99 998	28
33	7.98 223	1297	7.98 225	1297	2.01 775	9.99 998	27
34	7.99 520	1259	7.99 522	1259	2.00 478	9.99 998	26
35	8.00 779	1223	8.00 781	1223	1.99 219	9.99 998	25
36	8.02 002	1190	8.02 004	1190	1.97 996	9.99 998	24
37	8.03 192	1158	8.03 194	1159	1.96 806	9.99 997	23
38	8.04 350	1128	8.04 353	1128	1.95 647	9.99 997	22
39	8.05 478	1100	8.05 481	1100	1.94 519	9.99 997	21
40	8.06 578	1072	8.06 581	1072	1.93 419	9.99 997	20
41	8.07 650	1046	8.07 653	1047	1.92 347	9.99 997	19
42	8.08 696	1022	8.08 700	1022	1.91 300	9.99 997	18
43	8.09 718	999	8.09 722	998	1.90 278	9.99 997	17
44	8.10 717	976	8.10 720	976	1.89 280	9.99 996	16
45	8.11 693	954	8.11 696	955	1.88 304	9.99 996	15
46	8.12 647	934	8.12 651	934	1.87 349	9.99 996	14
47	8.13 581	914	8.13 585	915	1.86 415	9.99 996	13
48	8.14 495	896	8.14 500	895	1.85 500	9.99 996	12
49	8.15 391	877	8.15 395	878	1.84 605	9.99 996	11
50	8.16 268	860	8.16 273	860	1.83 727	9.99 995	10
51	8.17 128	843	8.17 133	843	1.82 867	9.99 995	9
52	8.17 971	827	8.17 976	828	1.82 024	9.99 995	8
53	8.18 798	812	8.18 804	812	1.81 196	9.99 995	7
54	8.19 610	797	8.19 616	797	1.80 384	9.99 995	6
55	8.20 407	782	8.20 413	782	1.79 587	9.99 994	5
56	8.21 189	769	8.21 195	769	1.78 805	9.99 994	4
57	8.21 958	755	8.21 964	756	1.78 036	9.99 994	3
58	8.22 713	743	8.22 720	742	1.77 280	9.99 994	2
59	8.23 456	.730	8.23 462	730	1.76 538	9.99 994	1
60	8.24 186		8.24 192		1.75 808	9.99 993	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	3476	3218	2997
.1	348	322	300
.2	695	644	599
.3	1043	965	899
.4	1390	1287	1199
.5	1738	1609	1498
	2802	2633	2483
.1	280	263	248
.2	560	527	497
.3	841	790	745
.4	1121	1053	993
.5	1401	1316	1242
	2227	2021	1848
.1	223	202	185
.2	445	404	370
.3	668	606	554
.4	891	808	739
.5	1113	1010	924
	1704	1579	1472
.1	170	158	147
.2	341	316	294
.3	511	474	442
.4	682	632	589
.5	852	789	736
	1379	1297	1223
.1	138	130	122
.2	276	259	245
.3	414	389	367
.4	552	519	489
.5	690	649	612
	1158	1100	1046
.1	116	110	105
.2	232	220	209
.3	347	330	314
.4	463	440	418
.5	579	550	523
	999	954	914
.1	100	95	91
.2	200	191	183
.3	300	286	274
.4	400	382	366
.5	500	477	457
	877	843	812
.1	88	84	81
.2	175	169	162
.3	263	253	244
.4	351	337	325
.5	438	422	406
	782	755	730
.1	78	75	73
.2	156	151	146
.3	235	226	219
.4	313	302	292
.5	391	377	365

Prop. Pts.

89°

1°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	8.24 186	717	8.24 192	718	1.75 808	9.99 993	60
1	8.24 903	706	8.24 910	706	1.75 090	9.99 993	59
2	8.25 609	695	8.25 616	696	1.74 384	9.99 993	58
3	8.26 304	684	8.26 312	684	1.73 688	9.99 993	57
4	8.26 988	673	8.26 996	673	1.73 004	9.99 992	56
5	8.27 661	663	8.27 669	663	1.72 331	9.99 992	55
6	8.28 324	653	8.28 332	654	1.71 668	9.99 992	54
7	8.28 977	644	8.28 986	643	1.71 014	9.99 992	53
8	8.29 621	634	8.29 629	634	1.70 371	9.99 992	52
9	8.30 255	624	8.30 263	625	1.69 737	9.99 991	51
10	8.30 879	616	8.30 888	617	1.69 112	9.99 991	50
11	8.31 495	608	8.31 505	607	1.68 495	9.99 991	49
12	8.32 103	599	8.32 112	599	1.67 888	9.99 990	48
13	8.32 702	590	8.32 711	591	1.67 289	9.99 990	47
14	8.33 292	583	8.33 302	584	1.66 698	9.99 990	46
15	8.33 875	575	8.33 886	575	1.66 114	9.99 990	45
16	8.34 450	568	8.34 461	568	1.65 539	9.99 989	44
17	8.35 018	560	8.35 029	561	1.64 971	9.99 989	43
18	8.35 578	553	8.35 590	553	1.64 410	9.99 989	42
19	8.36 131	547	8.36 143	546	1.63 857	9.99 989	41
20	8.36 678	539	8.36 689	540	1.63 311	9.99 988	40
21	8.37 217	533	8.37 229	533	1.62 771	9.99 988	39
22	8.37 750	526	8.37 762	527	1.62 238	9.99 988	38
23	8.38 276	520	8.38 289	520	1.61 711	9.99 987	37
24	8.38 796	514	8.38 809	514	1.61 191	9.99 987	36
25	8.39 310	508	8.39 323	509	1.60 677	9.99 987	35
26	8.39 818	502	8.39 832	502	1.60 168	9.99 986	34
27	8.40 320	496	8.40 334	496	1.59 666	9.99 986	33
28	8.40 816	491	8.40 830	491	1.59 170	9.99 986	32
29	8.41 307	485	8.41 321	486	1.58 679	9.99 985	31
30	8.41 792	480	8.41 807	480	1.58 193	9.99 985	30
31	8.42 272	474	8.42 287	475	1.57 713	9.99 985	29
32	8.42 746	470	8.42 762	470	1.57 238	9.99 984	28
33	8.43 216	464	8.43 232	464	1.56 768	9.99 984	27
34	8.43 680	459	8.43 696	460	1.56 304	9.99 984	26
35	8.44 139	455	8.44 156	455	1.55 844	9.99 983	25
36	8.44 594	450	8.44 611	450	1.55 389	9.99 983	24
37	8.45 044	445	8.45 061	446	1.54 939	9.99 983	23
38	8.45 489	441	8.45 507	441	1.54 493	9.99 982	22
39	8.45 930	436	8.45 948	437	1.54 052	9.99 982	21
40	8.46 366	433	8.46 385	432	1.53 615	9.99 982	20
41	8.46 799	427	8.46 817	428	1.53 183	9.99 981	19
42	8.47 226	424	8.47 245	424	1.52 755	9.99 981	18
43	8.47 650	419	8.47 669	420	1.52 331	9.99 981	17
44	8.48 069	416	8.48 089	416	1.51 911	9.99 980	16
45	8.48 485	411	8.48 505	412	1.51 495	9.99 980	15
46	8.48 896	408	8.48 917	408	1.51 083	9.99 979	14
47	8.49 304	404	8.49 325	404	1.50 675	9.99 979	13
48	8.49 708	400	8.49 729	401	1.50 271	9.99 979	12
49	8.50 108	396	8.50 130	397	1.49 870	9.99 978	11
50	8.50 504	393	8.50 527	393	1.49 473	9.99 978	10
51	8.50 897	390	8.50 920	390	1.49 080	9.99 977	9
52	8.51 287	386	8.51 310	386	1.48 690	9.99 977	8
53	8.51 673	382	8.51 696	383	1.48 304	9.99 977	7
54	8.52 055	379	8.52 079	380	1.47 921	9.99 976	6
55	8.52 434	376	8.52 459	376	1.47 541	9.99 976	5
56	8.52 810	373	8.52 835	373	1.47 165	9.99 975	4
57	8.53 183	369	8.53 208	370	1.46 792	9.99 975	3
58	8.53 552	367	8.53 578	367	1.46 422	9.99 974	2
59	8.53 919	363	8.53 945	363	1.46 055	9.99 974	1
60	8.54 282		8.54 308		1.45 692	9.99 974	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	717	695	673
.1	71.7	69.5	67.3
.2	143.4	139.0	134.6
.3	215.1	208.5	201.9
.4	286.8	278.0	269.2
.5	358.5	347.5	336.5

	653	634	616
.1	65.3	63.4	61.6
.2	130.6	126.8	123.2
.3	195.9	190.2	184.8
.4	261.2	253.6	246.4
.5	326.5	317.0	308.0

	599	583	568
.1	59.9	58.3	56.8
.2	119.8	116.6	113.6
.3	179.7	174.9	170.4
.4	239.6	233.2	227.2
.5	299.5	291.5	284.0

	553	539	526
.1	55.3	53.9	52.6
.2	110.6	107.8	105.2
.3	165.9	161.7	157.8
.4	221.2	215.6	210.4
.5	276.5	269.5	263.0

	514	502	490
.1	51.4	50.2	49
.2	102.8	100.4	98
.3	154.2	150.6	147
.4	205.6	200.8	196
.5	257.0	251.0	245

	480	470	460
.1	48	47	46
.2	96	94	92
.3	144	141	138
.4	192	188	184
.5	240	235	230

	450	440	430
.1	45	44	43
.2	90	88	86
.3	135	132	129
.4	180	176	172
.5	225	220	215

	420	410	400
.1	42	41	40
.2	84	82	80
.3	126	123	120
.4	168	164	160
.5	210	205	200

	390	380	370
.1	39	38	37
.2	78	76	74
.3	117	114	111
.4	156	152	148
.5	195	190	185

Prop. Pts.

88°

2°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	8.54 282	360	8.54 308	361	1.45 692	9.99 974	60
1	8.54 642	357	8.54 669	358	1.45 331	9.99 973	59
2	8.54 999	355	8.55 027	355	1.44 973	9.99 973	58
3	8.55 354	351	8.55 382	352	1.44 618	9.99 972	57
4	8.55 705	349	8.55 734	349	1.44 266	9.99 972	56
5	8.56 054	346	8.56 083	346	1.43 917	9.99 971	55
6	8.56 400	343	8.56 429	344	1.43 571	9.99 971	54
7	8.56 743	341	8.56 773	341	1.43 227	9.99 970	53
8	8.57 084	337	8.57 114	338	1.42 886	9.99 970	52
9	8.57 421	336	8.57 452	336	1.42 548	9.99 969	51
10	8.57 757	332	8.57 788	333	1.42 212	9.99 969	50
11	8.58 089	330	8.58 121	330	1.41 879	9.99 968	49
12	8.58 419	328	8 58 451	328	1.41 549	9.99 968	48
13	8.58 747	325	8.58 779	326	1.41 221	9.99 967	47
14	8.59 072	323	8.59 105	323	1.40 895̄	9.99 967	46
15	8.59 395̄	320	8.59 428	321	1.40 572	9.99 967	45
16	8.59 715	318	8.59 749	319	1.40 251	9.99 966	44
17	8.60 033	316	8.60 068	316	1.39 932	9.99 966	43
18	8.60 349	313	8.60 384	314	1.39 616	9.99 965̄	42
19	8.60 662	311	8.60 698	311	1.39 302	9.99 964	41
20	8.60 973	309	8.61 009	310	1.38 991	9.99 964	40
21	8.61 282	307	8.61 319	307	1.38 681	9.99 963	39
22	8.61 589	305	8.61 626	305	1.38 374	9.99 963	38
23	8.61 894	302	8.61 931	303	1.38 069	9.99 962	37
24	8.62 196	301	8.62 234	301	1.37 766	9.99 962	36
25	8.62 497	298	8.62 535	299	1.37 465̄	9.99 961	35
26	8.62 795̄	296	8.62 834	297	1.37 166	9.99 961	34
27	8.63 091	294	8.63 131	295	1.36 869	9.99 960	33
28	8.63 385	293	8.63 426	292	1.36 574	9.99 960	32
29	8.63 678	290	8.63 718	291	1.36 282	9.99 959	31
30	8.63 968	288	8.64 009	289	1.35 991	9.99 959	30
31	8.64 256	287	8.64 298	287	1.35 702	9.99 958	29
32	8.64 543	284	8.64 585	285	1.35 415̄	9.99 958	28
33	8.64 827	283	8.64 870	284	1.35 130	9.99 957	27
34	8.65 110	281	8.65 154	281	1.34 846	9.99 956	26
35	8.65 391	279	8.65 435	280	1.34 565̄	9.99 956	25
36	8.65 670	277	8.65 715̄	278	1.34 285	9.99 955	24
37	8.65 947	276	8.65 993	276	1.34 007	9.99 955̄	23
38	8.66 223	274	8.66 269	274	1.33 731	9.99 954	22
39	8.66 497	272	8.66 543	273	1.33 457	9.99 954	21
40	8.66 769	270	8.66 816	271	1.33 184	9.99 953	20
41	8.67 039	269	8.67 087	269	1.32 913	9.99 952	19
42	8.67 308	267	8.67 356	268	1.32 644	9.99 952	18
43	8.67 575	266	8.67 624	266	1.32 376	9.99 951	17
44	8.67 841	263	8.67 890	264	1.32 110	9.99 951	16
45	8.68 104	263	8.68 154	263	1.31 846	9.99 95̄0	15
46	8.68 367	260	8.68 417	261	1.31 583	9.99 949	14
47	8.68 627	259	8.68 678	260	1.31 322	9.99 949	13
48	8.68 886	258	8.68 938	258	1.31 062	9.99 948	12
49	8.69 144	256	8.69 196	257	1.30 804	9.99 948	11
50	8.69 400	254	8.69 453	255	1.30 547	9.99 947	10
51	8.69 654	253	8.69 708	254	1.30 292	9.99 946	9
52	8.69 907	252	8.69 962	252	1.30 038	9.99 946	8
53	8.70 159	250	8.70 214	251	1.29 786	9.99 945̄	7
54	8.70 409	249	8.70 465̄	249	1.29 535	9.99 944	6
55	8.70 658	247	8.70 714	248	1.29 286	9.99 944	5
56	8.70 905̄	246	8.70 962	246	1.29 038	9.99 943	4
57	8.71 151	244	8.71 208	245	1.28 792	9.99 942	3
58	8.71 395	243	8.71 453	244	1.28 547	9.99 942	2
59	8.71 638	242	8.71 697	243	1.28 303	9.99 941	1
60	8.71 880		8.71 940		1.28 060	9.99 940	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.			
	360	350	340
.1	36	35	34
.2	72	70	68
.3	108	105	102
.4	144	140	136
.5	180	175	170
.6	216	210	204
.7	252	245	238
.8	288	280	272
.9	324	315	306
	330	320	310
.1	33	32	31
.2	66	64	62
.3	99	96	93
.4	132	128	124
.5	165	160	155
.6	198	192	186
.7	231	224	217
.8	264	256	248
.9	297	288	279
	300	290	285
.1	30	29	28.5
.2	60	58	57.0
.3	90	87	85.5
.4	120	116	114.0
.5	150	145	142.5
.6	180	174	171.0
.7	210	203	199.5
.8	240	232	228.0
.9	270	261	256.5
	280	275	270
.1	28.0	27.5	27.0
.2	56.0	55.0	54.0
.3	84.0	82.5	81.0
.4	112.0	110.0	108.0
.5	140.0	137.5	135.0
.6	168.0	165.0	162.0
.7	196.0	192.5	189.0
.8	224.0	220.0	216.0
.9	252.0	247.5	243.0
	265	260	255
.1	.26.5	.26.0	.25.5
.2	.53.0	.52.0	.51.0
.3	.79.5	.78.0	.76.5
.4	106.0	104.0	102.0
.5	132.5	130.0	127.5
.6	159.0	156.0	153.0
.7	185.5	182.0	178.5
.8	212.0	208.0	204.0
.9	238.5	234.0	229.5
	250	245	240
.1	.25.0	.24.5	.24.0
.2	.50.0	.49.0	.48.0
.3	.75.0	.73.5	.72.0
.4	100.0	198.0	.96.0
.5	125.0	122.5	120.0
.6	150.0	147.0	144.0
.7	175.0	171.5	168.0
.8	200.0	196.0	192.0
.9	225.0	220.5	216.0
Prop. Pts.			

87°

3°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	8.71 880	240	8.71 940	241	1.28 060	9.99 940	60
1	8.72 120	239	8.72 181	239	1.27 819	9.99 940	59
2	8.72 359	238	8.72 420	239	1.27 580	9.99 939	58
3	8.72 597	237	8.72 659	237	1.27 341	9.99 938	57
4	8.72 834	235	8.72 896	236	1.27 104	9.99 938	56
5	8.73 069	234	8.73 132	234	1.26 868	9.99 937	55
6	8.73 303	232	8.73 366	234	1.26 634	9.99 936	54
7	8.73 535	232	8.73 600	232	1.26 400	9.99 936	53
8	8.73 767	230	8.73 832	231	1.26 168	9.99 935	52
9	8.73 997	229	8.74 063	229	1.25 937	9.99 934	51
10	8.74 226	228	8.74 292	229	1.25 708	9.99 934	50
11	8.74 454	226	8.74 521	227	1.25 479	9.99 933	49
12	8.74 680	226	8.74 748	226	1.25 252	9.99 932	48
13	8.74 906	224	8.74 974	225	1.25 026	9.99 932	47
14	8.75 130	223	8.75 199	224	1.24 801	9.99 931	46
15	8.75 353	222	8.75 423	222	1.24 577	9.99 930	45
16	8.75 575	220	8.75 645	222	1.24 355	9.99 929	44
17	8.75 795	220	8.75 867	220	1.24 133	9.99 929	43
18	8.76 015	219	8.76 087	219	1.23 913	9.99 928	42
19	8.76 234	217	8.76 306	219	1.23 694	9.99 927	41
20	8.76 451	216	8.76 525	217	1.23 475	9.99 926	40
21	8.76 667	216	8.76 742	216	1.23 258	9.99 926	39
22	8.76 883	214	8.76 958	215	1.23 042	9.99 925	38
23	8.77 097	213	8.77 173	214	1.22 827	9.99 924	37
24	8.77 310	212	8.77 387	213	1.22 613	9.99 923	36
25	8.77 522	211	8.77 600	211	1.22 400	9.99 923	35
26	8.77 733	210	8.77 811	211	1.22 189	9.99 922	34
27	8.77 943	209	8.78 022	210	1.21 978	9.99 921	33
28	8.78 152	208	8.78 232	209	1.21 768	9.99 920	32
29	8.78 360	208	8.78 441	208	1.21 559	9.99 920	31
30	8.78 568	206	8.78 649	206	1.21 351	9.99 919	30
31	8.78 774	205	8.78 855	206	1.21 145	9.99 918	29
32	8.78 979	204	8.79 061	205	1.20 939	9.99 917	28
33	8.79 183	203	8.79 266	204	1.20 734	9 99 917	27
34	8.79 386	202	8.79 470	203	1.20 530	9.99 916	26
35	8.79 588	201	8.79 673	202	1.20 327	9.99 915	25
36	8.79 789	201	8.79 875	201	1.20 125	9.99 914	24
37	8.79 990	199	8.80 076	201	1.19 924	9.99 913	23
38	8.80 189	199	8.80 277	199	1.19 723	9.99 913	22
39	8.80 388	197	8.80 476	198	1.19 524	9.99 912	21
40	8.80 585	197	8.80 674	198	1.19 326	9.99 911	20
41	8.80 782	196	8.80 872	196	1.19 128	9.99 910	19
42	8.80 978	195	8.81 068	196	1.18 932	9.99 909	18
43	8.81 173	194	8.81 264	195	1.18 736	9.99 909	17
44	8.81 367	193	8.81 459	194	1.18 541	9.99 908	16
45	8.81 560	192	8.81 653	193	1.18 347	9.99 907	15
46	8.81 752	192	8.81 846	192	1.18 154	9.99 906	14
47	8.81 944	190	8.82 038	192	1.17 962	9.99 905	13
48	8.82 134	190	8.82 230	190	1.17 770	9.99 904	12
49	8.82 324	189	8.82 420	190	1.17 580	9.99 904	11
50	8.82 513	188	8.82 610	189	1.17 390	9.99 903	10
51	8.82 701	187	8.82 799	188	1.17 201	9.99 902	9
52	8.82 888	187	8.82 987	188	1.17 013	9.99 901	8
53	8.83 075	186	8.83 175	186	1.16 825	9.99 900	7
54	8.83 261	185	8.83 361	186	1.16 639	9.99 899	6
55	8.83 446	184	8.83 547	185	1.16 453	9.99 898	5
56	8.83 630	183	8.83 732	184	1.16 268	9.99 898	4
57	8.83 813	183	8.83 916	184	1.16 084	9.99 897	3
58	8.83 996	181	8.84 100	182	1.15 900	9.99 896	2
59	8.84 177	181	8.84 282	182	1.15 718	9.99 895	1
60	8.84 358		8.84 464		1.15 536	9.99 894	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	238	234	229
.1	23.8	23.4	22.9
.2	47.6	46.8	45.8
.3	71.4	70.2	68.7
.4	95.2	93.6	91.6
.5	119.0	117.0	114.5
.6	142.8	140.4	137.4
.7	166.6	163.8	160.3
.8	190.4	187.2	183.2
.9	214.2	210.6	206.1

	225	220	216
.1	22.5	22.0	21.6
.2	45.0	44.0	43.2
.3	67.5	66.0	64.8
.4	90.0	88.0	86.4
.5	112.5	110.0	108.0
.6	135.0	132.0	129.6
.7	157.5	154.0	151.2
.8	180.0	176.0	172.8
.9	202.5	198.0	194.4

	212	208	204
.1	21.2	20.8	20.4
.2	42.4	41.6	40.8
.3	63.6	62.4	61.2
.4	84.8	83.2	81.6
.5	106.0	104.0	102.0
.6	127.2	124.8	122.4
.7	148.4	145.6	142.8
.8	169.6	166.4	163.2
.9	190.8	187.2	183.6

	201	197	193
.1	20.1	19.7	19.3
.2	40.2	39.4	38.6
.3	60.3	59.1	57.9
.4	80.4	78.8	77.2
.5	100.5	98.5	96.5
.6	120.6	118.2	115.8
.7	140.7	137.9	135.1
.8	160.8	157.6	154.4
.9	180.9	177.3	173.7

	189	185	181
.1	18.9	18.5	18.1
.2	37.8	37.0	36.2
.3	56.7	55.5	54.3
.4	75.6	74.0	72.4
.5	94.5	92.5	90.5
.6	113.4	111.0	108.6
.7	132.3	129.5	126.7
.8	151.2	148.0	144.8
.9	170.1	166.5	162.9

	4	3	2	1
.1	0.4	0.3	0.2	0.1
.2	0.8	0.6	0.4	0.2
.3	1.2	0.9	0.6	0.3
.4	1.6	1.2	0.8	0.4
.5	2.0	1.5	1.0	0.5
.6	2.4	1.8	1.2	0.6
.7	2.8	2.1	1.4	0.7
.8	3.2	2.4	1.6	0.8
.9	3.6	2.7	1.8	0.9

Prop. Pts.

86°

4°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	8.84 358	181	8.84 464	182	1.15 536	9.99 894	60
1	8.84 539	179	8.84 646	180	1.15 354	9.99 893	59
2	8.84 718	179	8.84 826	180	1.15 174	9.99 892	58
3	8.84 897	178	8.85 006	179	1.14 994	9.99 891	57
4	8.85 075	177	8.85 18$\bar{5}$	178	1.14 815	9.99 891	56
5	8.85 252	177	8.85 363	177	1.14 637	9.99 890	55
6	8.85 429	176	8.85 540	177	1.14 460	9.99 889	54
7	8.85 60$\bar{5}$	175	8.85 717	176	1.14 283	9.99 888	53
8	8.85 780	175	8.85 893	176	1.14 107	9.99 887	52
9	8.85 95$\bar{5}$	173	8.86 069	174	1.13 931	9.99 886	51
10	8.86 128	173	8.86 243	174	1.13 757	9.99 885	50
11	8.86 301	173	8.86 417	174	1.13 583	9.99 884	49
12	8.86 474	171	8.86 591	172	1.13 409	9.99 883	48
13	8.86 645	171	8.86 763	172	1.13 237	9.99 882	47
14	8.86 816	171	8.86 935	171	1.13 06$\bar{5}$	9.99 881	46
15	8.86 987	169	8.87 106	171	1.12 894	9.99 880	45
16	8.87 156	169	8.87 277	170	1.12 723	9.99 879	44
17	8.87 325	169	8.87 447	169	1.12 553	9.99 879	43
18	8.87 494	167	8.87 616	169	1.12 384	9.99 878	42
19	8.87 661	168	8.87 78$\bar{5}$	168	1.12 215	9.99 877	41
20	8.87 829	166	8.87 953	167	1.12 047	9.99 876	40
21	8.87 99$\bar{5}$	166	8.88 120	167	1.11 880	9.99 87$\bar{5}$	39
22	8.88 161	165	8.88 287	166	1.11 713	9.99 874	38
23	8.88 326	164	8.88 453	165	1.11 547	9.99 873	37
24	8.88 490	164	8.88 618	165	1.11 382	9.99 872	36
25	8.88 654	163	8.88 783	165	1.11 217	9.99 871	35
26	8.88 817	163	8.88 948	163	1.11 052	9.99 870	34
27	8.88 980	162	8.89 111	163	1.10 889	9.99 869	33
28	8.89 142	162	8.89 274	163	1.10 726	9.99 868	32
29	8.89 304	160	8.89 437	161	1.10 563	9.99 867	31
30	8.89 464	161	8.89 598	162	1.10 402	9.99 866	30
31	8.89 62$\bar{5}$	159	8.89 760	160	1.10 240	9.99 86$\bar{5}$	29
32	8.89 784	159	8.89 920	160	1.10 080	9.99 864	28
33	8.89 943	159	8.90 080	160	1.09 920	9.99 863	27
34	8.90 102	158	8.90 240	159	1.09 760	9.99 862	26
35	8.90 260	157	8.90 399	158	1.09 601	9.99 861	25
36	8.90 417	157	8.90 557	158	1.09 443	9.99 860	24
37	8.90 574	156	8.90 71$\bar{5}$	157	1.09 285	9.99 859	23
38	8.90 730	155	8.90 872	157	1.09 128	9.99 858	22
39	8.90 885	155	8.91 029	156	1.08 971	9.99 857	21
40	8.91 040	155	8.91 18$\bar{5}$	155	1.08 815	9.99 856	20
41	8.91 19$\bar{5}$	154	8.91 340	155	1.08 660	9.99 85$\bar{5}$	19
42	8.91 349	153	8.91 495	155	1.08 50$\bar{5}$	9.99 854	18
43	8.91 502	153	8.91 650	153	1.08 350	9.99 853	17
44	8.91 655	152	8.91 803	154	1.08 197	9.99 852	16
45	8.91 807	152	8.91 957	153	1.08 043	9.99 851	15
46	8.91 959	151	8.92 110	152	1.07 890	9.99 850	14
47	8.92 110	151	8.92 262	152	1.07 738	9.99 848	13
48	8.92 261	150	8.92 414	151	1.07 586	9.99 847	12
49	8.92 411	150	8.92 56$\bar{5}$	151	1.07 435	9.99 846	11
50	8.92 561	149	8.92 716	150	1.07 284	9.99 845	10
51	8.92 710	149	8.92 866	150	1.07 134	9.99 844	9
52	8.92 859	148	8.93 016	149	1.06 984	9.99 843	8
53	8.93 007	147	8.93 16$\bar{5}$	148	1.06 835	9.99 842	7
54	8.93 154	147	8.93 313	149	1.06 687	9.99 841	6
55	8.93 301	147	8.93 462	147	1.06 538	9.99 840	5
56	8.93 448	146	8 93 609	147	1.06 391	9.99 839	4
57	8.93 594	146	8.93 756	147	1.06 244	9.99 838	3
58	8.93 740	145	8.93 903	146	1.06 097	9.99 837	2
59	8.93 88$\bar{5}$	145	8.94 049	146	1.05 951	9.99 836	1
60	8.94 030		8.94 195		1.05 805	9.99 834	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	181	179	177
.1	18.1	17.9	17.7
.2	36.2	35.8	35.4
.3	54.3	53.7	53.1
.4	72.4	71.6	70.8
.5	90.5	89.5	88.5
.6	108.6	107.4	106.2
.7	126.7	125.3	123.9
.8	144.8	143.2	141.6
.9	162.9	161.1	159.3
	175	**173**	**171**
.1	17.5	17.3	17.1
.2	35.0	34.6	34.2
.3	52.5	51.9	51.3
.4	70.0	69.2	68.4
.5	87.5	86.5	85.5
.6	105.0	103.8	102.6
.7	122.5	121.1	119.7
.8	140.0	138.4	136.8
.9	157.5	155.7	153.9
	168	**166**	**164**
.1	.16.8	16.6	16.4
.2	33.6	33.2	32.8
.3	50.4	49.8	49.2
.4	67.2	66.4	65.6
.5	84.0	83.0	82 0
.6	100.8	99.6	98.4
.7	117.6	116.2	114.8
.8	134.4	132.8	131.2
.9	151.2	149.4	147.6
	162	**159**	**157**
.1	16.2	15.9	15.7
.2	32.4	31.8	31.4
.3	48.6	47.7	47.1
.4	64.8	63.6	62.8
.5	81.0	79.5	78.5
.6	97.2	95.4	94.2
.7	113.4	111.3	109.9
.8	129.6	127.2	125.6
.9	145.8	143.1	141.3
	155	**153**	**151**
.1	15.5	15.3	15.1
.2	31.0	30.6	30.2
.3	46.5	45.9	45.3
.4	62.0	61.2	60.4
.5	77.5	76.5	75.5
.6	93.0	91.8	90.6
.7	108.5	107.1	105.7
.8	124.0	122.4	120.8
.9	139.5	137.7	135.9
	149	**147**	**1**
.1	14.9	14.7	0.1
.2	29.8	29.4	0.2
.3	44.7	44.1	0.3
.4	59.6	58.8	0.4
.5	74.5	73.5	0.5
.6	89.4	88.2	0.6
.7	104.3	102.9	0.7
.8	119.2	117.6	0.8
.9	134.1	132.3	0.9

Prop. Pts.

85°

5°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	8.94 030	144	8.94 195	145	1.05 805̄	9.99 834	60
1	8.94 174	143	8.94 340	145	1.05 660	9.99 833	59
2	8.94 317	144	8.94 485	145	1.05 515̄	9.99 832	58
3	8.94 461	142	8.94 630	143	1.05 370	9.99 831	57
4	8.94 603	143	8.94 773	144	1.05 227	9.99 830	56
5	8.94 746	141	8.94 917	143	1.05 083	9.99 829	55
6	8.94 887	142	8.95 060	142	1.04 940	9.99 828	54
7	8.95 029	141	8.95 202	142	1.04 798	9.99 827	53
8	8.95 170	140	8.95 344	142	1.04 656	9.99 825	52
9	8.95 310	140	8.95 486	141	1.04 514	9.99 824	51
10	8.95 45̄0	139	8.95 627	140	1.04 373	9.99 823	50
11	8.95 589	139	8.95 767	141	1.04 233	9.99 822	49
12	8.95 728	139	8.95 908	139	1.04 092	9.99 821	48
13	8.95 867	138	8.96 047	140	1.03 953	9.99 820	47
14	8.96 005	138	8.96 187	138	1.03 813	9.99 819	46
15	8.96 143	137	8.96 325	139	1.03 675̄	9.99 817	45
16	8.96 280	137	8.96 464	138	1.03 536	9.99 816	44
17	8.96 417	136	8.96 602	137	1.03 398	9.99 815	43
18	8.96 553	136	8.96 739	138	1.03 261	9.99 814	42
19	8.96 689	136	8.96 877	136	1.03 123	9.99 813	41
20	8.96 825̄	135	8.97 013	137	1.02 987	9.99 812	40
21	8.96 960	135	8.97 15̄0	135	1.02 850	9.99 810	39
22	8.97 095̄	134	8.97 285	136	1.02 715̄	9.99 809	38
23	8.97 229	134	8.97 421	135	1.02 579	9.99 808	37
24	8.97 363	133	8.97 556	135	1.02 444	9.99 807	36
25	8.97 496	133	8.97 691	134	1.02 309	9.99 806	35
26	8.97 629	133	8.97 825̄	134	1.02 175	9.99 804	34
27	8.97 762	132	8.97 959	133	1.02 041	9.99 803	33
28	8.97 894	132	8.98 092	133	1.01 908	9.99 802	32
29	8.98 026	131	8.98 225	133	1.01 775̄	9.99 801	31
30	8.98 157	131	8.98 358	132	1.01 642	9.99 800	30
31	8.98 288	131	8.98 490	132	1.01 510	9.99 798	29
32	8.98 419	130	8.98 622	131	1.01 378	9.99 797	28
33	8.98 549	130	8.98 753	131	1.01 247	9.99 796	27
34	8.98 679	129	8.98 884	131	1.01 116	9.99 795̄	26
35	8.98 808	129	8.99 015̄	130	1.00 985	9.99 793	25
36	8.98 937	129	8.99 145	130	1.00 855̄	9.99 792	24
37	8.99 066	128	8.99 275	130	1.00 725̄	9.99 791	23
38	8.99 194	128	8.99 405̄	129	1.00 595	9.99 790	22
39	8.99 322	128	8.99 534	128	1.00 466	9.99 788	21
40	8.99 45̄0	127	8.99 662	129	1.00 338	9.99 787	20
41	8.99 577	127	8.99 791	128	1.00 209	9.99 786	19
42	8.99 704	126	8.99 919	127	1.00 081	9.99 785̄	18
43	8.99 830	126	9.00 046	128	0.99 954	9.99 783	17
44	8.99 956	126	9.00 174	127	0.99 826	9.99 782	16
45	9.00 082	125	9.00 301	126	0.99 699	9.99 781	15
46	9.00 207	125	9.00 427	126	0.99 573	9.99 780	14
47	9.00 332	124	9.00 553	126	0.99 447	9.99 778	13
48	9.00 456	125	9.00 679	126	0.99 321	9.99 777	12
49	9.00 581	123	9.00 805̄	125	0.99 195	9.99 776	11
50	9.00 704	124	9.00 930	125	0.99 070	9.99 775̄	10
51	9.00 828	123	9.01 055̄	124	0.98 945	9.99 773	9
52	9.00 951	123	9.01 179	124	0.98 821	9.99 772	8
53	9.01 074	122	9.01 303	124	0.98 697	9.99 771	7
54	9.01 196	122	9.01 427	123	0.98 573	9.99 769	6
55	9.01 318	122	9.01 550	123	0.98 45̄0	9.99 768	5
56	9.01 440	121	9.01 673	123	0.98 327	9.99 767	4
57	9.01 561	121	9.01 796	122	0.98 204	9.99 765	3
58	9.01 682	121	9.01 918	122	0.98 082	9.99 764	2
59	9.01 803	120	9.02 040	122	0.97 960	9.99 763	1
60	9.01 923		9.02 162		0.97 838	9.99 761	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	145	143	141
.1	14.5	14.3	14.1
.2	29.0	28.6	28.2
.3	43.5	42.9	42.3
.4	58.0	57.2	56.4
.5	72.5	71.5	70.5
.6	87.0	85.8	84.6
.7	101.5	100.1	98.7
.8	116.0	114.4	112.8
.9	130.5	128.7	126.9
	139	**138**	**136**
.1	13.9	13.8	13.6
.2	27.8	27.6	27.2
.3	41.7	41.4	40.8
.4	55.6	55.2	54.4
.5	69.5	69.0	68.0
.6	83.4	82.8	81.6
.7	97.3	96.6	95.2
.8	111.2	110.4	108.8
.9	125.1	124.2	122.4
	135	**133**	**131**
.1	13.5	13.3	13.1
.2	27.0	26.6	26.2
.3	40.5	39.9	39.3
.4	54.0	53.2	52.4
.5	67.5	66.5	65.5
.6	81.0	79.8	78.6
.7	94.5	93.1	91.7
.8	108.0	106.4	104.8
.9	121.5	119.7	117.9
	129	**128**	**126**
.1	12.9	12.8	12.6
.2	25.8	25.6	25.2
.3	38.7	38.4	37.8
.4	51.6	51.2	50.4
.5	64.5	64.0	63.0
.6	77.4	76.8	75.6
.7	90.3	89.6	88.2
.8	103.2	102.4	100.8
.9	116.1	115.2	113.4
	125	**123**	**122**
.1	12.5	12.3	12.2
.2	25.0	24.6	24.4
.3	37.5	36.9	36.6
.4	50.0	49.2	48.8
.5	62.5	61.5	61.0
.6	75.0	73.8	73.2
.7	87.5	86.1	85.4
.8	100.0	98.4	97.6
.9	112.5	110.7	109.8
	121	**120**	**1**
.1	12.1	12.0	0.1
.2	24.2	24.0	0.2
.3	36.3	36.0	0.3
.4	48.4	48.0	0.4
.5	60.5	60.0	0.5
.6	72.6	72.0	0.6
.7	84.7	84.0	0.7
.8	96.8	96.0	0.8
.9	108.9	108.0	0.9

Prop. Pts.

84°

6°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.01 923	120	9.02 162	121	0.97 838	9.99 761	60
1	9.02 043	120	9.02 283	121	0.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	0.97 596	9.99 759	58
3	9.02 283	119	9.02 525	120	0.97 475	9.99 757	57
4	9.02 402	118	9.02 645	121	0.97 355	9.99 756	56
5	9.02 520	119	9.02 766	119	0.97 234	9.99 755	55
6	9.02 639	118	9.02 885	120	0.97 115	9.99 753	54
7	9.02 757	117	9.03 005	119	0.96 995	9.99 752	53
8	9.02 874	118	9.03 124	118	0.96 876	9.99 751	52
9	9.02 992	117	9.03 242	119	0.96 758	9.99 749	51
10	9.03 109	117	9.03 361	118	0.96 639	9.99 748	50
11	9.03 226	116	9.03 479	118	0.96 521	9.99 747	49
12	9.03 342	116	9.03 597	117	0.96 403	9.99 745	48
13	9.03 458	116	9.03 714	118	0.96 286	9.99 744	47
14	9.03 574	116	9.03 832	116	0.96 168	9.99 742	46
15	9.03 690	115	9.03 948	117	0.96 052	9.99 741	45
16	9.03 805	115	9.04 065	116	0.95 935	9.99 740	44
17	9.03 920	114	9.04 181	116	0.95 819	9.99 738	43
18	9.04 034	115	9.04 297	116	0.95 703	9.99 737	42
19	9.04 149	113	9.04 413	115	0.95 587	9.99 736	41
20	9.04 262	114	9.04 528	115	0.95 472	9.99 734	40
21	9.04 376	114	9.04 643	115	0.95 357	9.99 733	39
22	9.04 490	113	9.04 758	115	0.95 242	9.99 731	38
23	9.04 603	112	9.04 873	114	0.95 127	9.99 730	37
24	9.04 715	113	9.04 987	114	0.95 013	9.99 728	36
25	9.04 828	112	9.05 101	113	0.94 899	9.99 727	35
26	9.04 940	112	9.05 214	114	0.94 786	9.99 726	34
27	9.05 052	112	9.05 328	113	0.94 672	9.99 724	33
28	9.05 164	111	9.05 441	112	0.94 559	9.99 723	32
29	9.05 275	111	9.05 553	113	0.94 447	9.99 721	31
80	9.05 386	111	9.05 666	112	0.94 334	9.99 720	80
31	9.05 497	110	9.05 778	112	0.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	0.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	0.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	0.93 887	9.99 714	26
35	9.05 937	109	9.06 224	111	0.93 776	9.99 713	25
36	9.06 046	109	9.06 335	110	0.93 665	9.99 711	24
37	9.06 155	109	9.06 445	111	0.93 555	9.99 710	23
38	9.06 264	108	9.06 556	110	0.93 444	9.99 708	22
39	9.06 372	109	9.06 666	109	0.93 334	9.99 707	21
40	9.06 481	108	9.06 775	110	0.93 225	9.99 705	20
41	9.06 589	107	9.06 885	109	0.93 115	9.99 704	19
42	9.06 696	108	9.06 994	109	0.93 006	9.99 702	18
43	9.06 804	107	9.07 103	108	0.92 897	9.99 701	17
44	9.06 911	107	9.07 211	109	0.92 789	9.99 699	16
45	9.07 018	106	9.07 320	108	0.92 680	9.99 698	15
46	9.07 124	107	9.07 428	108	0.92 572	9.99 696	14
47	9.07 231	106	9.07 536	107	0.92 464	9.99 695	13
48	9.07 337	105	9.07 643	108	0.92 357	9.99 693	12
49	9.07 442	106	9.07 751	107	0.92 249	9.99 692	11
50	9.07 548	105	9.07 858	106	0.92 142	9.99 690	10
51	9.07 653	105	9.07 964	107	0.92 036	9.99 689	9
52	9.07 758	105	9.08 071	106	0.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	0.91 823	9.99 686	7
54	9.07 968	104	9.08 283	106	0.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	0.91 611	9.99 683	5
56	9.08 176	104	9.08 495	105	0.91 505	9.99 681	4
57	9.08 280	103	9.08 600	105	0.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	0.91 295	9.99 678	2
59	9.08 486	103	9.08 810	104	0.91 190	9.99 677	1
60	9.08 589		9.08 914		0.91 086	9.99 675	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

83°

Prop. Pts.

	121	120	119
.1	12.1	12.0	11.9
.2	24.2	24.0	23.8
.3	36.3	36.0	35.7
.4	48.4	48.0	47.6
.5	60.5	60.0	59.5
.6	72.6	72.0	71.4
.7	84.7	84.0	83.3
.8	96.8	96.0	95.2
.9	108.9	108.0	107.1
	118	117	116
.1	11.8	11.7	11.6
.2	23.6	23.4	23.2
.3	35.4	35.1	34.8
.4	47.2	46.8	46.4
.5	59.0	58.5	58.0
.6	70.8	70.2	69.6
.7	82.6	81.9	81.2
.8	94.4	93.6	92.8
.9	106.2	105.3	104.4
	115	114	113
.1	11.5	11.4	11.3
.2	23.0	22.8	22.6
.3	34.5	34.2	33.9
.4	46.0	45.6	45.2
.5	57.5	57.0	56.5
.6	69.0	68.4	67.8
.7	80.5	79.8	79.1
.8	92.0	91.2	90.4
.9	103.5	102.6	101.7
	112	111	110
.1	11.2	11.1	11.0
.2	22.4	22.2	22.0
.3	33.6	33.3	33.0
.4	44.8	44.4	44.0
.5	56.0	55.5	55.0
.6	67.2	66.6	66.0
.7	78.4	77.7	77.0
.8	89.6	88.8	88.0
.9	100.8	99.9	99.0
	109	108	107
.1	10.9	10.8	10.7
.2	21.8	21.6	21.4
.3	32.7	32.4	32.1
.4	43.6	43.2	42.8
.5	54.5	54.0	53.5
.6	65.4	64.8	64.2
.7	76.3	75.6	74.9
.8	87.2	86.4	85.6
.9	98.1	97.2	96.3
	106	105	104
.1	10.6	10.5	10.4
.2	21.2	21.0	20.8
.3	31.8	31.5	31.2
.4	42.4	42.0	41.6
.5	53.0	52.5	52.0
.6	63.6	63.0	62.4
.7	74.2	73.5	72.8
.8	84.8	84.0	83.2
.9	95.4	94.5	93.6

Prop. Pts.

7°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.08 589	103	9.08 914	105	0.91 086	9.99 675	**60**
1	9.08 692	103	9.09 019	104	0.90 981	9.99 674	59
2	9.08 795̄	102	9.09 123	104	0.90 877	9.99 672	58
3	9.08 897	102	9.09 227	103	0.90 773	9.99 670	57
4	9.08 999	102	9.09 330	104	0.90 670	9.99 669	56
5	9.09 101	101	9.09 434	103	0.90 566	9.99 667	55
6	9.09 202	102	9.09 537	103	0.90 463	9.99 666	54
7	9.09 304	101	9.09 640	102	0.90 360	9.99 664	53
8	9.09 405̄	101	9.09 742	103	0.90 258	9.99 663	52
9	9.09 506	100	9.09 845̄	102	0.90 155	9.99 661	51
10	9.09 606	101	9.09 947	102	0.90 053	9.99 659	**50**
11	9.09 707	100	9.10 049	101	0.89 951	9.99 658	49
12	9.09 807	100	9.10 150	102	0.89 850	9.99 656	48
13	9.09 907	99	9.10 252	101	0.89 748	9.99 655̄	47
14	9.10 006	100	9.10 353	101	0.89 647	9.99 653	46
15	9.10 106	99	9.10 454	101	0.89 546	9.99 651	45
16	9.10 205̄	99	9.10 555̄	101	0.89 445	9.99 650	44
17	9.10 304	98	9.10 656	100	0.89 344	9.99 648	43
18	9.10 402	99	9.10 756	100	0.89 244	9.99 647	42
19	9.10 501	98	9.10 856	100	0.89 144	9.99 645̄	41
20	9.10 599	98	9.10 956	100	0.89 044	9.99 643	**40**
21	9.10 697	98	9.11 056	99	0.88 944	9.99 642	39
22	9.10 795	98	9.11 155	99	0.88 845̄	9.99 640	38
23	9.10 893	97	9.11 254	99	0.88 746	9.99 638	37
24	9.10 990	97	9.11 353	99	0.88 647	9.99 637	36
25	9.11 087	97	9.11 452	99	0.88 548	9.99 635	35
26	9.11 184	97	9.11 551	98	0.88 449	9.99 633	34
27	9.11 281	96	9.11 649	98	0.88 351	9.99 632	33
28	9.11 377	97	9.11 747	98	0.88 253	9.99 630	32
29	9.11 474	96	9.11 845	98	0.88 155̄	9.99 629	31
30	9.11 570	96	9.11 943	97	0.88 057	9.99 627	**30**
31	9.11 666	95	9.12 040	98	0.87 960	9.99 625	29
32	9.11 761	96	9.12 138	97	0.87 862	9.99 624	28
33	9.11 857	95	9.12 235̄	97	0.87 765	9.99 622	27
34	9.11 952	95	9.12 332	96	0.87 668	9.99 620	26
35	9.12 047	95	9.12 428	97	0.87 572	9.99 618	25
36	9.12 142	94	9.12 525̄	96	0.87 475	9.99 617	24
37	9.12 236	95	9.12 621	96	0.87 379	9.99 615	23
38	9.12 331	94	9.12 717	96	0.87 283	9.99 613	22
39	9.12 425̄	94	9.12 813	96	0.87 187	9.99 612	21
40	9.12 519	93	9.12 909	95	0.87 091	9.99 610	**20**
41	9.12 612	94	9.13 004	95	0.86 996	9.99 608	19
42	9.12 706	93	9.13 099	95	0.86 901	9.99 607	18
43	9.12 799	93	9.13 194	95	0.86 806	9.99 605̄	17
44	9.12 892	93	9.13 289	95	0.86 711	9.99 603	16
45	9.12 985	93	9.13 384	94	0.86 616	9.99 601	15
46	9.13 078	93	9.13 478	95	0.86 522	9.99 600	14
47	9.13 171	92	9.13 573	94	0.86 427	9.99 598	13
48	9.13 263	92	9.13 667	94	0.86 333	9.99 596	12
49	9.13 355	92	9.13 761	93	0.86 239	9.99 595̄	11
50	9.13 447	92	9.13 854	94	0.86 146	9.99 593	**10**
51	9.13 539	91	9.13 948	93	0.86 052	9.99 591	9
52	9.13 630	92	9.14 041	93	0.85 959	9.99 589	8
53	9.13 722	91	9.14 134	93	0.85 866	9.99 588	7
54	9.13 813	91	9.14 227	93	0.85 773	9.99 586	6
55	9.13 904	90	9.14 320	92	0.85 680	9.99 584	5
56	9.13 994	91	9.14 412	92	0.85 588	9.99 582	4
57	9.14 085	90	9.14 504	93	0.85 496	9.99 581	3
58	9,14 175	91	9.14 597	91	0.85 403	9.99 579	2
59	9.14 266	90	9.14 688	92	0.85 312	9.99 577	1
60	9.14 356		9.14 780		0.85 220	9.99 575	**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

82°

Prop. Pts.

	105	104	103
.1	10.5	10.4	10.3
.2	21.0	20.8	20.6
.3	31.5	31.2	30.9
.4	42.0	41.6	41.2
.5	52.5	52.0	51.5
.6	63.0	62.4	61.8
.7	73.5	72.8	72.1
.8	84.0	83.2	82.4
.9	94.5	93.6	92.7

	102	101	100
.1	10.2	10.1	10.0
.2	20.4	20.2	20.0
.3	30.6	30.3	30.0
.4	40.8	40.4	40.0
.5	51.0	50.5	50.0
.6	61.2	60.6	60.0
.7	71.4	70.7	70.0
.8	81.6	80.8	80.0
.9	91.8	90.9	90.0

	99	98
.1	9.9	9.8
.2	19.8	19.6
.3	29.7	29.4
.4	39.6	39.2
.5	49.5	49.0
.6	59.4	58.8
.7	69.3	68.6
.8	79.2	78.4
.9	89.1	88.2

	97	96	95
.1	9.7	9.6	9.5
.2	19.4	19.2	19.0
.3	29.1	28.8	28.5
.4	38.8	38.4	38.0
.5	48.5	48.0	47.5
.6	58.2	57.6	57.0
.7	67.9	67.2	66.5
.8	77.6	76.8	76.0
.9	87.3	86.4	85.5

	94	93	92
.1	9.4	9.3	9.2
.2	18.8	18.6	18.4
.3	28.2	27.9	27.6
.4	37.6	37.2	36.8
.5	47.0	46.5	46.0
.6	56.4	55.8	55.2
.7	65.8	65.1	64.4
.8	75.2	74.4	73.6
.9	84.6	83.7	82.8

	91	90	2
.1	9.1	9.0	0.2
.2	18.2	18.0	0.4
.3	27.3	27.0	0.6
.4	36.4	36.0	0.8
.5	45.5	45.0	1.0
.6	54.6	54.0	1.2
.7	63.7	63.0	1.4
.8	72.8	72.0	1.6
.9	81.9	81.0	1.8

Prop. Pts.

8°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.14 356	89	9.14 780	92	0.85 220	9.99 575	60
1	9.14 445	90	9.14 872	91	0.85 128	9.99 574	59
2	9.14 535	89	9.14 963	91	0.85 037	9.99 572	58
3	9.14 624	90	9.15 054	91	0.84 946	9.99 570	57
4	9.14 714	89	9.15 145	91	0.84 855	9.99 568	56
5	9.14 803	88	9.15 236	91	0.84 764	9.99 566	55
6	9.14 891	89	9.15 327	90	0.84 673	9.99 565	54
7	9.14 980	89	9.15 417	91	0.84 583	9.99 563	53
8	9.15 069	88	9.15 508	90	0.84 492	9.99 561	52
9	9.15 157	88	9.15 598	90	0.84 402	9.99 559	51
10	9.15 245	88	9.15 688	89	0.84 312	9.99 557	50
11	9.15 333	88	9.15 777	90	0.84 223	9.99 556	49
12	9.15 421	87	9.15 867	89	0.84 133	9.99 554	48
13	9.15 508	88	9.15 956	90	0.84 044	9.99 552	47
14	9.15 596	87	9.16 046	89	0.83 954	9.99 550	46
15	9.15 683	87	9.16 135	89	0.83 865	9.99 548	45
16	9.15 770	87	9.16 224	88	0.83 776	9.99 546	44
17	9.15 857	87	9.16 312	89	0.83 688	9.99 545	43
18	9.15 944	86	9.16 401	88	0.83 599	9.99 543	42
19	9.16 030	86	9.16 489	88	0.83 511	9.99 541	41
20	9.16 116	87	9.16 577	88	0.83 423	9.99 539	40
21	9.16 203	86	9.16 665	88	0.83 335	9.99 537	39
22	9.16 289	85	9.16 753	88	0.83 247	9.99 535	38
23	9.16 374	86	9.16 841	87	0.83 159	9.99 533	37
24	9.16 460	85	9.16 928	88	0.83 072	9.99 532	36
25	9.16 545	86	9.17 016	87	0.82 984	9.99 530	35
26	9.16 631	85	9.17 103	87	0.82 897	9.99 528	34
27	9.16 716	85	9.17 190	87	0.82 810	9.99 526	33
28	9.16 801	85	9.17 277	86	0.82 723	9.99 524	32
29	9.16 886	84	9.17 363	87	0.82 637	9.99 522	31
30	9.16 970	85	9.17 450	86	0.82 550	9.99 520	30
31	9.17 055	84	9.17 536	86	0.82 464	9.99 518	29
32	9.17 139	84	9.17 622	86	0.82 378	9.99 517	28
33	9.17 223	84	9.17 708	86	0.82 292	9.99 515	27
34	9.17 307	84	9.17 794	86	0.82 206	9.99 513	26
35	9.17 391	83	9.17 880	85	0.82 120	9.99 511	25
36	9.17 474	84	9.17 965	86	0.82 035	9.99 509	24
37	9.17 558	83	9.18 051	85	0.81 949	9.99 507	23
38	9.17 641	83	9.18 136	85	0.81 864	9.99 505	22
39	9.17 724	83	9.18 221	85	0.81 779	9.99 503	21
40	9.17 807	83	9.18 306	85	0.81 694	9.99 501	20
41	9.17 890	83	9.18 391	84	0.81 609	9.99 499	19
42	9.17 973	82	9.18 475	85	0.81 525	9.99 497	18
43	9.18 055	82	9.18 560	84	0.81 440	9.99 495	17
44	9.18 137	83	9.18 644	84	0.81 356	9.99 494	16
45	9.18 220	82	9.18 728	84	0.81 272	9.99 492	15
46	9.18 302	81	9.18 812	84	0.81 188	9.99 490	14
47	9.18 383	82	9.18 896	83	0.81 104	9.99 488	13
48	9.18 465	82	9.18 979	84	0.81 021	9.99 486	12
49	9.18 547	81	9.19 063	83	0.80 937	9.99 484	11
50	9.18 628	81	9.19 146	83	0.80 854	9.99 482	10
51	9.18 709	81	9.19 229	83	0.80 771	9.99 480	9
52	9.18 790	81	9.19 312	83	0.80 688	9.99 478	8
53	9.18 871	81	9.19 395	83	0.80 605	9.99 476	7
54	9.18 952	81	9.19 478	83	0.80 522	9.99 474	6
55	9.19 033	80	9.19 561	82	0.80 439	9.99 472	5
56	9.19 113	80	9.19 643	82	0.80 357	9.99 470	4
57	9.19 193	80	9.19 725	82	0.80 275	9.99 468	3
58	9.19 273	80	9.19 807	82	0.80 193	9.99 466	2
59	9.19 353	80	9.19 889	82	0.80 111	9.99 464	1
60	9.19 433		9.19 971		0.80 029	9.99 462	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

81°

Prop. Pts.

	92	91	90
.1	9.2	9.1	9.0
.2	18.4	18.2	18.0
.3	27.6	27.3	27.0
.4	36.8	36.4	36.0
.5	46.0	45.5	45.0
.6	55.2	54.6	54.0
.7	64.4	63.7	63.0
.8	73.6	72.8	72.0
.9	82.8	81.9	81.0

	89	88
.1	8.9	8.8
.2	17.8	17.6
.3	26 7	26.4
.4	35.6	35.2
.5	44.5	44.0
.6	53.4	52.8
.7	62.3	61.6
.8	71.2	70.4
.9	80.1	79.2

	87	86
.1	8.7	8.6
.2	17.4	17.2
.3	26.1	25.8
.4	34.8	34.4
.5	43.5	43.0
.6	52.2	51.6
.7	60.9	60.2
.8	69.6	68.8
.9	78.3	77.4

	85	84
.1	8.5	8.4
.2	17.0	16.8
.3	25.5	25.2
.4	34.0	33.6
.5	42.5	42.0
.6	51.0	50.4
.7	59.5	58.8
.8	68.0	67.2
.9	76.5	75.6

	83	82
.1	8.3	8.2
.2	16.6	16.4
.3	24.9	24.6
.4	33.2	32.8
.5	41.5	41.0
.6	49.8	49.2
.7	58.1	57.4
.8	66.4	65.6
.9	74.7	73.8

	81	80	2
.1	8.1	8.0	0.2
.2	16.2	16.0	0.4
.3	24.3	24.0	0.6
.4	32.4	32.0	0.8
.5	40.5	40.0	1.0
.6	48.6	48.0	1.2
.7	56.7	56.0	1.4
.8	64.8	64.0	1.6
.9	72.9	72.0	1.8

Prop. Pts.

9°

'	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.19 433	80	9.19 971	82	0.80 029	9.99 462	60
1	9.19 513	79	9.20 053	81	0.79 947	9.99 460	59
2	9.19 592	80	9.20 134	82	0.79 866	9.99 458	58
3	9.19 672	79	9.20 216	81	0.79 784	9.99 456	57
4	9.19 751	79	9.20 297	81	0.79 703	9.99 454	56
5	9.19 830	79	9.20 378	81	0.79 622	9.99 452	55
6	9.19 909	79	9.20 459	81	0.79 541	9.99 450	54
7	9.19 988	79	9.20 540	81	0.79 460	9.99 448	53
8	9.20 067	78	9.20 621	80	0.79 379	9.99 446	52
9	9.20 145	78	9.20 701	81	0.79 299	9.99 444	51
10	9.20 223	79	9.20 782	80	0.79 218	9.99 442	50
11	9.20 302	78	9.20 862	80	0.79 138	9.99 440	49
12	9.20 380	78	9.20 942	80	0.79 058	9.99 438	48
13	9.20 458	77	9.21 022	80	0.78 978	9.99 436	47
14	9.20 535	78	9.21 102	80	0.78 898	9.99 434	46
15	9.20 613	78	9.21 182	79	0.78 818	9.99 432	45
16	9.20 691	77	9.21 261	80	0.78 739	9.99 429	44
17	9.20 768	77	9.21 341	79	0.78 659	9.99 427	43
18	9.20 845	77	9.21 420	79	0.78 580	9.99 425	42
19	9.20 922	77	9.21 499	79	0.78 501	9.99 423	41
20	9.20 999	77	9.21 578	79	0.78 422	9.99 421	40
21	9.21 076	77	9.21 657	79	0.78 343	9.99 419	39
22	9.21 153	76	9.21 736	78	0.78 264	9.99 417	38
23	9.21 229	77	9.21 814	79	0.78 186	9.99 415	37
24	9.21 306	76	9.21 893	78	0.78 107	9.99 413	36
25	9.21 382	76	9.21 971	78	0.78 029	9.99 411	35
26	9.21 458	76	9.22 049	78	0.77 951	9.99 409	34
27	9.21 534	76	9.22 127	78	0.77 873	9.99 407	33
28	9.21 610	75	9.22 205	78	0.77 795	9.99 404	32
29	9.21 685	76	9.22 283	78	0.77 717	9.99 402	31
30	9.21 761	75	9.22 361	77	0.77 639	9.99 400	30
31	9.21 836	76	9.22 438	78	0.77 562	9.99 398	29
32	9.21 912	75	9.22 516	77	0.77 484	9.99 396	28
33	9.21 987	75	9.22 593	77	0.77 407	9.99 394	27
34	9.22 062	75	9.22 670	77	0.77 330	9.99 392	26
35	9.22 137	74	9.22 747	77	0.77 253	9 99 390	25
36	9.22 211	75	9.22 824	77	0.77 176	9.99 388	24
37	9.22 286	75	9.22 901	76	0.77 099	9.99 385	23
38	9.22 361	74	9.22 977	77	0.77 023	9.99 383	22
39	9.22 435	74	9.23 054	76	0.76 946	9.99 381	21
40	9.22 509	74	9.23 130	76	0.76 870	9.99 379	20
41	9.22 583	74	9.23 206	77	0.76 794	9.99 377	19
42	9.22 657	74	9.23 283	76	0.76 717	9.99 375	18
43	9.22 731	74	9.23 359	76	0.76 641	9.99 372	17
44	9.22 805	73	9.23 435	75	0.76 565	9.99 370	16
45	9.22 878	74	9.23 510	76	0.76 490	9.99 368	15
46	9.22 952	73	9.23 586	75	0.76 414	9.99 366	14
47	9.23 025	73	9.23 661	76	0.76 339	9.99 364	13
48	9.23 098	73	9.23 737	75	0.76 263	9.99 362	12
49	9.23 171	73	9.23 812	75	0.76 188	9.99 359	11
50	9.23 244	73	9.23 887	75	0.76 113	9.99 357	10
51	9.23 317	73	9.23 962	75	0.76 038	9.99 355	9
52	9.23 390	72	9.24 037	75	0.75 963	9.99 353	8
53	9.23 462	73	9.24 112	74	0.75 888	9.99 351	7
54	9.23 535	72	9.24 186	75	0.75 814	9.99 348	6
55	9.23 607	72	9.24 261	74	0.75 739	9.99 346	5
56	9.23 679	73	9.24 335	75	0.75 665	9.99 344	4
57	9.23 752	71	9.24 410	74	0.75 590	9.99 342	3
58	9.23 823	72	9.24 484	74	0.75 516	9.99 340	2
59	9.23 895	72	9.24 558	74	0.75 442	9.99 337	1
60	9.23 967		9.24 632		0.75 368	9.99 335	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	'

80°

Prop. Pts.

	82	81	80
.1	8.2	8.1	8.0
.2	16.4	16.2	16.0
.3	24.6	24.3	24.0
.4	32.8	32.4	32.0
.5	41.0	40.5	40.0
.6	49.2	48.6	48.0
.7	57.4	56.7	56.0
.8	65.6	64.8	64.0
.9	73.8	72.9	72.0

	79	78
.1	7.9	7.8
.2	15.8	15.6
.3	23.7	23.4
.4	31.6	31.2
.5	39.5	39.0
.6	47.4	46.8
.7	55.3	54.6
.8	63.2	62.4
.9	71.1	70.2

	77	76
.1	7.7	7.6
.2	15.4	15.2
.3	23.1	22.8
.4	30.8	30.4
.5	38.5	38.0
.6	46.2	45.6
.7	53.9	53.2
.8	61.6	60.8
.9	69.3	68.4

	75	74
.1	7.5	7.4
.2	15.0	14.8
.3	22.5	22.2
.4	30.0	29.6
.5	37.5	37.0
.6	45.0	44.4
.7	52.5	51.8
.8	60.0	59.2
.9	67.5	66.6

	73	72
.1	7.3	7.2
.2	14.6	14.4
.3	21.9	21.6
.4	29.2	28.8
.5	36.5	36.0
.6	43.8	43.2
.7	51.1	50.4
.8	58.4	57.6
.9	65.7	64.8

	71	3	2
.1	7.1	0.3	0.2
.2	14.2	0.6	0.4
.3	21.3	0.9	0.6
.4	28.4	1.2	0.8
.5	35.5	1.5	1.0
.6	42.6	1.8	1.2
.7	49.7	2.1	1.4
.8	56.8	2.4	1.6
.9	63.9	2.7	1.8

Prop. Pts.

10°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.23 967	72	9.24 632	74	0.75 368	9.99 335	60
1	9.24 039	71	9.24 706	73	0.75 294	9.99 333	59
2	9.24 110	71	9.24 779	74	0.75 221	9.99 331	58
3	9.24 181	72	9.24 853	73	0.75 147	9.99 328	57
4	9.24 253	71	9.24 926	74	0.75 074	9.99 326	56
5	9.24 324	71	9.25̄ 000	73	0.75 000	9.99 324	55
6	9.24 395̄	71	9.25 073	73	0.74 927	9.99 322	54
7	9.24 466	70	9.25 146	73	0.74 854	9.99 319	53
8	9.24 536	71	9.25 219	73	0.74 781	9.99 317	52
9	9.24 607	70	9.25 292	73	0.74 708	9.99 315̄	51
10	9.24 677	71	9.25 365̄	72	0.74 635	9.99 313	50
11	9.24 748	70	9.25 437	73	0.74 563	9.99 310	49
12	9.24 818	70	9.25 510	72	0.74 490	9.99 308	48
13	9.24 888	70	9.25 582	73	0.74 418	9.99 306	47
14	9.24 958	70	9.25 655̄	72	0.74 345	9.99 304	46
15	9.25 028	70	9.25 727	72	0.74 273	9.99 301	45
16	9.25 098	70	9.25 799	72	0.74 201	9.99 299	44
17	9.25 168	69	9.25 871	72	0.74 129	9.99 297	43
18	9.25 237	70	9.25 943	72	0.74 057	9.99 294	42
19	9.25 307	69	9.26 015̄	71	0.73 985	9.99 292	41
20	9.25 376	69	9.26 086	72	0.73 914	9.99 290	40
21	9.25 445	69	9.26 158	71	0.73 842	9.99 288	39
22	9.25 514	69	9.26 229	72	0.73 771	9.99 285	38
23	9.25 583	69	9.26 301	71	0.73 699	9.99 283	37
24	9.25 652	69	9.26 372	71	0.73 628	9.99 281	36
25	9.25 721	69	9.26 443	71	0.73 557	9.99 278	35
26	9.25 790	68	9.26 514	71	0.73 486	9.99 276	34
27	9.25 858	69	9.26 585̄	70	0.73 415̄	9.99 274	33
28	9.25 927	68	9.26 655	71	0.73 345̄	9.99 271	32
29	9.25 995	68	9.26 726	71	0.73 274	9.99 269	31
30	9.26 063	68	9.26 797	70	0.73 203	9.99 267	30
31	9.26 131	68	9.26 867	70	0.73 133	9.99 264	29
32	9.26 199	68	9.26 937	71	0.73 063	9.99 262	28
33	9.26 267	68	9.27 008	70	0.72 992	9.99 260	27
34	9.26 335	68	9.27 078	70	0.72 922	9.99 257	26
35	9.26 403	67	9.27 148	70	0.72 852	9.99 255̄	25
36	9.26 470	68	9.27 218	70	0.72 782	9.99 252	24
37	9.26 538	67	9.27 288	69	0.72 712	9.99 250	23
38	9.26 605	67	9.27 357	70	0.72 643	9.99 248	22
39	9.26 672	67	9.27 427	69	0.72 573	9.99 245	21
40	9.26 739	67	9.27 496	70	0.72 504	9.99 243	20
41	9.26 806	67	9.27 566	69	0.72 434	9.99 241	19
42	9.26 873	67	9.27 635	69	0.72 365̄	9.99 238	18
43	9.26 940	67	9.27 704	69	0.72 296	9.99 236	17
44	9.27 007	66	9.27 773	69	0.72 227	9.99 233	16
45	9.27 073	67	9.27 842	69	0.72 158	9.99 231	15
46	9.27 140	66	9.27 911	69	0.72 089	9.99 229	14
47	9.27 206	67	9.27 980	69	0.72 020	9.99 226	13
48	9.27 273	66	9.28 049	68	0.71 951	9.99 224	12
49	9.27 339	66	9.28 117	69	0.71 883	9.99 221	11
50	9.27 405̄	66	9.28 186	68	0.71 814	9.99 219	10
51	9.27 471	66	9.28 254	69	0.71 746	9.99 217	9
52	9.27 537	65	9.28 323	68	0.71 677	9.99 214	8
53	9.27 602	66	9.28 391	68	0.71 609	9.99 212	7
54	9.27 668	66	9.28 459	68	0.71 541	9.99 209	6
55	9.27 734	65	9.28 527	68	0.71 473	9.99 207	5
56	9.27 799	65	9.28 595̄	67	0.71 405	9.99 204	4
57	9.27 864	66	9.28 662	68	0.71 338	9.99 202	3
58	9.27 930	65	9.28 730	68	0.71 270	9.99 200	2
59	9.27 995̄	65	9.28 798	67	0.71 202	9.99 197	1
60	9.28 060		9.28 865		0.71 135̄	9.99 195̄	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

79°

Prop. Pts.

	74	73	72	71	70	69	68	67	66	65	3	2
.1	7.4	7.3	7.2	7.1	7.0	6.9	6.8	6.7	6.6	6.5	0.3	0.2
.2	14.8	14.6	14.4	14.2	14.0	13.8	13.6	13.4	13.2	13.0	0.6	0.4
.3	22.2	21.9	21.6	21.3	21.0	20.7	20.4	20.1	19.8	19.5	0.9	0.6
.4	29.6	29.2	28.8	28.4	28.0	27.6	27.2	26.8	26.4	26.0	1.2	0.8
.5	37.0	36.5	36.0	35.5	35.0	34.5	34.0	33.5	33.0	32.5	1.5	1.0
.6	44.4	43.8	43.2	42.6	42.0	41.4	40.8	40.2	39.6	39.0	1.8	1.2
.7	51.8	51.1	50.4	49.7	49.0	48.3	47.6	46.9	46.2	45.5	2.1	1.4
.8	59.2	58.4	57.6	56.8	56.0	55.2	54.4	53.6	52.8	52.0	2.4	1.6
.9	66.6	65.7	64.8	63.9	63.0	62.1	61.2	60.3	59.4	58.5	2.7	1.8

Prop. Pts.

11°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.28 060	65	9.28 865	68	0.71 135̄	9.99 195̄	**60**
1	9.28 125̄	65	9.28 933	67	0.71 067	9.99 192	59
2	9.28 190	64	9.29 000	67	0.71 000	9.99 190	58
3	9.28 254	65	9.29 067	67	0.70 933	9.99 187	57
4	9.28 319	65	9.29 134	67	0.70 866	9.99 185̄	56
5	9.28 384	64	9.29 201	67	0.70 799	9.99 182	55
6	9.28 448	64	9.29 268	67	0.70 732	9.99 180	54
7	9.28 512	65	9.29 335̄	67	0.70 665	9.99 177	53
8	9.28 577	64	9.29 402	66	0.70 598	9.99 175̄	52
9	9.28 641	64	9.29 468	67	0.70 532	9.99 172	51
10	9.28 705̄	64	9.29 535̄	66	0.70 465	9.99 170	**50**
11	9.28 769	64	9.29 601	67	0.70 399	9.99 167	49
12	9.28 833	63	9.29 668	66	0.70 332	9.99 165̄	48
13	9.28 896	64	9.29 734	66	0.70 266	9.99 162	47
14	9.28 960	64	9.29 800	66	0.70 200	9.99 160	46
15	9.29 024	63	9.29 866	66	0.70 134	9.99 157	45
16	9.29 087	63	9.29 932	66	0.70 068	9.99 155̄	44
17	9.29 150	64	9.29 998	66	0.70 002	9.99 152	43
18	9.29 214	63	9.30 064	66	0.69 936	9.99 15̄0	42
19	9.29 277	63	9.30 130	65	0.69 870	9.99 147	41
20	9.29 340	63	9.30 195	66	0.69 805̄	9.99 145̄	**40**
21	9.29 403	63	9.30 261	65	0.69 739	9.99 142	39
22	9.29 466	63	9.30 326	65	0.69 674	9.99 140	38
23	9.29 529	62	9.30 391	66	0.69 609	9.99 137	37
24	9.29 591	63	9.30 457	65	0.69 543	9.99 135̄	36
25	9.29 654	62	9.30 522	65	0.69 478	9.99 132	35
26	9.29 716	63	9.30 587	65	0.69 413	9.99 130	34
27	9.29 779	62	9.30 652	65	0.69 348	9.99 127	33
28	9.29 841	62	9.30 717	65	0.69 283	9.99 124	32
29	9.29 903	63	9.30 782	64	0.69 218	9.99 122	31
30	9.29 966	62	9.30 846	65	0.69 154	9.99 119	**30**
31	9.30 028	62	9.30 911	64	0.69 089	9.99 117	29
32	9.30 090	61	9.30 975	65	0.69 025̄	9.99 114	28
33	9.30 151	62	9.31 040	64	0.68 960	9.99 112	27
34	9.30 213	62	9.31 104	64	0.68 896	9.99 109	26
35	9.30 275̄	61	9.31 168	65	0.68 832	9.99 106	25
36	9.30 336	62	9.31 233	64	0.68 767	9.99 104	24
37	9.30 398	61	9.31 297	64	0.68 703	9.99 101	23
38	9.30 459	62	9.31 361	64	0.68 639	9.99 099	22
39	9.30 521	61	9.31 425̄	64	0.68 575	9.99 096	21
40	9.30 582	61	9.31 489	63	0.68 511	9.99 093	**20**
41	9.30 643	61	9.31 552	64	0.68 448	9.99 091	19
42	9.30 704	61	9.31 616	63	0.68 384	9.99 088	18
43	9.30 765	61	9.31 679	64	0.68 321	9.99 086	17
44	9.30 826	61	9.31 743	63	0.68 257	9.99 083	16
45	9.30 887	60	9.31 806	64	0.68 194	9.99 080	15
46	9.30 947	61	9.31 870	63	0.68 130	9.99 078	14
47	9.31 008	60	9.31 933	63	0.68 067	9.99 075	13
48	9.31 068	61	9.31 996	63	0.68 004	9.99 072	12
49	9.31 129	60	9.32 059	63	0.67 941	9.99 070	11
50	9.31 189	61	9.32 122	63	0.67 878	9.99 067	**10**
51	9.31 25̄0	60	9.32 185	63	0.67 815̄	9.99 064	9
52	9.31 310	60	9.32 248	63	0.67 752	9.99 062	8
53	9.31 370	60	9.32 311	62	0.67 689	9.99 059	7
54	9.31 430	60	9.32 373	63	0.67 627	9.99 056	6
55	9.31 490	59	9.32 436	62	0.67 564	9.99 054	5
56	9.31 549	60	9.32 498	63	0.67 502	9.99 051	4
57	9.31 609	60	9.32 561	62	0.67 439	9.99 048	3
58	9.31 669	59	9.32 623	62	0.67 377	9.99 046	2
59	9.31 728	60	9.32 685	62	0.67 315̄	9.99 043	1
60	9.31 788		9.32 747		0.67 253	9.99 040	**0**
	L. Cos.	**d.**	**L. Cotg.**	**c. d.**	**L. Tang.**	**L. Sin.**	**′**

78°

Prop. Pts.

	68	67
.1	6.8	6.7
.2	13.6	13.4
.3	20.4	20.1
.4	27.2	26.8
.5	34.0	33.5
.6	40.8	40.2
.7	47.6	46.9
.8	54.4	53.6
.9	61.2	60.3
	66	**65**
.1	6.6	6.5
.2	13.2	13.0
.3	19.8	19.5
.4	26.4	26.0
.5	33.0	32.5
.6	39.6	39.0
.7	46.2	45.5
.8	52.8	52.0
.9	59.4	58.5
	64	**63**
.1	6.4	6.3
.2	12.8	12.6
.3	19.2	18.9
.4	25.6	25.2
.5	32.0	31.5
.6	38.4	37.8
.7	44.8	44.1
.8	51.2	50.4
.9	57.6	56.7
	62	**61**
.1	6.2	6.1
.2	12.4	12.2
.3	18.6	18.3
.4	24.8	24.4
.5	31.0	30.5
.6	37.2	36.6
.7	43.4	42.7
.8	49.6	48.8
.9	55.8	54.9
	60	**59**
.1	6.0	5.9
.2	12.0	11.8
.3	18.0	17.7
.4	24.0	23.6
.5	30.0	29.5
.6	36.0	35.4
.7	42.0	41.3
.8	48.0	47.2
.9	54.0	53.1
	3	**2**
.1	0.3	0.2
.2	0.6	0.4
.3	0.9	0.6
.4	1.2	0.8
.5	1.5	1.0
.6	1.8	1.2
.7	2.1	1.4
.8	2.4	1.6
.9	2.7	1.8

Prop. Pts.

12°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.31 788	59	9.32 747	63	0.67 253	9.99 040	60
1	9.31 847	60	9.32 810	62	0.67 190	9.99 038	59
2	9.31 907	59	9.32 872	61	0.67 128	9.99 035	58
3	9.31 966	59	9.32 933	62	0.67 067	9.99 032	57
4	9.32 025	59	9.32 995	62	0.67 005	9.99 030	56
5	9.32 084	59	9.33 057	62	0.66 943	9.99 027	55
6	9.32 143	59	9.33 119	61	0.66 881	9.99 024	54
7	9.32 202	59	9.33 180	62	0.66 820	9.99 022	53
8	9.32 261	58	9.33 242	61	0.66 758	9.99 019	52
9	9.32 319	59	9.33 303	62	0.66 697	9.99 016	51
10	9.32 378	59	9.33 365	61	0.66 635	9.99 013	50
11	9.32 437	58	9.33 426	61	0.66 574	9.99 011	49
12	9.32 495	58	9.33 487	61	0.66 513	9.99 008	48
13	9.32 553	59	9.33 548	61	0.66 452	9.99 005	47
14	9.32 612	58	9.33 609	61	0.66 391	9.99 002	46
15	9.32 670	58	9.33 670	61	0.66 330	9.99 000	45
16	9.32 728	58	9.33 731	61	0.66 269	9.98 997	44
17	9.32 786	58	9.33 792	61	0.66 208	9.98 994	43
18	9.32 844	58	9.33 853	60	0.66 147	9.98 991	42
19	9.32 902	58	9.33 913	61	0.66 087	9.98 989	41
20	9.32 960	58	9.33 974	60	0.66 026	9.98 986	40
21	9.33 018	57	9.34 034	61	0.65 966	9.98 983	39
22	9.33 075	58	9.34 095	60	0.65 905	9.98 980	38
23	9.33 133	57	9.34 155	60	0.65 845	9.98 978	37
24	9.33 190	58	9.34 215	61	0.65 785	9.98 975	36
25	9.33 248	57	9.34 276	60	0.65 724	9.98 972	35
26	9.33 305	57	9.34 336	60	0.65 664	9.98 969	34
27	9.33 362	58	9.34 396	60	0.65 604	9.98 967	33
28	9.33 420	57	9.34 456	60	0.65 544	9.98 964	32
29	9.33 477	57	9.34 516	60	0.65 484	9.98 961	31
30	9.33 534	57	9.34 576	59	0.65 424	9.98 958	30
31	9.33 591	56	9.34 635	60	0.65 365	9.98 955	29
32	9.33 647	57	9.34 695	60	0.65 305	9.98 953	28
33	9.33 704	57	9.34 755	59	0.65 245	9.98 950	27
34	9.33 761	57	9.34 814	60	0.65 186	9.98 947	26
35	9.33 818	56	9.34 874	59	0.65 126	9.98 944	25
36	9.33 874	57	9.34 933	59	0.65 067	9.98 941	24
37	9.33 931	56	9.34 992	59	0.65 008	9.98 938	23
38	9.33 987	56	9.35 051	60	0.64 949	9.98 936	22
39	9.34 043	57	9.35 111	59	0.64 889	9.98 933	21
40	9.34 100	56	9.35 170	59	0.64 830	9.98 930	20
41	9.34 156	56	9.35 229	59	0.64 771	9.98 927	19
42	9.34 212	56	9.35 288	59	0.64 712	9.98 924	18
43	9.34 268	56	9.35 347	58	0.64 653	9.98 921	17
44	9.34 324	56	9.35 405	59	0.64 595	9.98 919	16
45	9.34 380	56	9.35 464	59	0.64 536	9.98 916	15
46	9.34 436	55	9.35 523	58	0.64 477	9.98 913	14
47	9.34 491	56	9.35 581	59	0.64 419	9.98 910	13
48	9.34 547	55	9.35 640	58	0.64 360	9.98 907	12
49	9.34 602	56	9.35 698	59	0.64 302	9.98 904	11
50	9.34 658	55	9.35 757	58	0.64 243	9.98 901	10
51	9.34 713	56	9.35 815	58	0.64 185	9.98 898	9
52	9.34 769	55	9.35 873	58	0.64 127	9.98 896	8
53	9.34 824	55	9.35 931	58	0.64 069	9.98 893	7
54	9.34 879	55	9.35 989	58	0.64 011	9.98 890	6
55	9.34 934	55	9.36 047	58	0.63 953	9.98 887	5
56	9.34 989	55	9.36 105	58	0.63 895	9.98 884	4
57	9.35 044	55	9.36 163	58	0.63 837	9.98 881	3
58	9.35 099	55	9.36 221	58	0.63 779	9.98 878	2
59	9.35 154	55	9.36 279	57	0.63 721	9.98 875	1
60	9.35 209		9.36 336		0.63 664	9.98 872	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	63	62
.1	6.3	6.2
.2	12.6	12.4
.3	18.9	18.6
.4	25.2	24.8
.5	31.5	31.0
.6	37.8	37.2
.7	44.1	43.4
.8	50.4	49.6
.9	56.7	55.8

	61	60
.1	6.1	6.0
.2	12.2	12.0
.3	18.3	18.0
.4	24.4	24.0
.5	30.5	30.0
.6	36.6	36.0
.7	42.7	42.0
.8	48.8	48.0
.9	54.9	54.0

	59
.1	5.9
.2	11.8
.3	17.7
.4	23.6
.5	29.5
.6	35.4
.7	41.3
.8	47.2
.9	53.1

	58	57
.1	5.8	5.7
.2	11.6	11.4
.3	17.4	17.1
.4	23.2	22.8
.5	29.0	28.5
.6	34.8	34.2
.7	40.6	39.9
.8	46.4	45.6
.9	52.2	51.3

	56	55
.1	5.6	5.5
.2	11.2	11.0
.3	16.8	16.5
.4	22.4	22.0
.5	28.0	27.5
.6	33.6	33.0
.7	39.2	38.5
.8	44.8	44.0
.9	50.4	49.5

	3	2
.1	0.3	0.2
.2	0.6	0.4
.3	0.9	0.6
.4	1.2	0.8
.5	1.5	1.0
.6	1.8	1.2
.7	2.1	1.4
.8	2.4	1.6
.9	2.7	1.8

Prop. Pts.

77°

13°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.35 209	54	9.36 336	58	0.63 664	9.98 872	60
1	9.35 263	55	9.36 394	58	0.63 606	9.98 869	59
2	9.35 318	55	9.36 452	57	0.63 548	9.98 867	58
3	9.35 373	54	9.36 509	57	0.63 491	9.98 864	57
4	9.35 427	54	9.36 566	58	0.63 434	9.98 861	56
5	9.35 481	55	9.36 624	57	0.63 376	9.98 858	55
6	9.35 536	54	9.36 681	57	0.63 319	9.98 855	54
7	9.35 590	54	9.36 738	57	0.63 262	9.98 852	53
8	9.35 644	54	9.36 795	57	0.63 205	9.98 849	52
9	9.35 698	54	9.36 852	57	0.63 148	9.98 846	51
10	9.35 752	54	9.36 909	57	0.63 091	9.98 843	50
11	9.35 806	54	9.36 966	57	0.63 034	9.98 840	49
12	9.35 860	54	9.37 023	57	0.62 977	9.98 837	48
13	9.35 914	54	9.37 080	57	0.62 920	9.98 834	47
14	9.35 968	54	9.37 137	56	0.62 863	9.98 831	46
15	9.36 022	53	9.37 193	57	0.62 807	9.98 828	45
16	9.36 075	54	9.37 250	56	0.62 750	9.98 825	44
17	9.36 129	53	9.37 306	57	0.62 694	9.98 822	43
18	9.36 182	54	9.37 363	56	0.62 637	9.98 819	42
19	9.36 236	53	9.37 419	57	0.62 581	9.98 816	41
20	9.36 289	53	9.37 476	56	0.62 524	9.98 813	40
21	9.36 342	53	9.37 532	56	0.62 468	9.98 810	39
22	9.36 395	54	9.37 588	56	0.62 412	9.98 807	38
23	9.36 449	53	9.37 644	56	0.62 356	9.98 804	37
24	9.36 502	53	9.37 700	56	0.62 300	9.98 801	36
25	9.36 555	53	9.37 756	56	0.62 244	9.98 798	35
26	9.36 608	52	9.37 812	56	0.62 188	9.98 795	34
27	9.36 660	53	9.37 868	56	0.62 132	9.98 792	33
28	9.36 713	53	9.37 924	56	0.62 076	9.98 789	32
29	9.36 766	53	9.37 980	55	0.62 020	9.98 786	31
30	9.36 819	52	9.38 035	56	0.61 965	9.98 783	30
31	9.36 871	53	9.38 091	56	0.61 909	9.98 780	29
32	9.36 924	52	9.38 147	55	0.61 853	9.98 777	28
33	9.36 976	52	9.38 202	55	0.61 798	9.98 774	27
34	9.37 028	53	9.38 257	56	0.61 743	9.98 771	26
35	9.37 081	52	9.38 313	55	0.61 687	9.98 768	25
36	9.37 133	52	9.38 368	55	0.61 632	9.98 765	24
37	9.37 185	52	9.38 423	56	0.61 577	9.98 762	23
38	9.37 237	52	9.38 479	55	0.61 521	9.98 759	22
39	9.37 289	52	9.38 534	55	0.61 466	9.98 756	21
40	9.37 341	52	9.38 589	55	0.61 411	9.98 753	20
41	9.37 393	52	9.38 644	55	0.61 356	9.98 750	19
42	9.37 445	52	9.38 699	55	0.61 301	9.98 746	18
43	9.37 497	52	9.38 754	54	0.61 246	9.98 743	17
44	9.37 549	51	9.38 808	55	0.61 192	9.98 740	16
45	9.37 600	52	9.38 863	55	0.61 137	9.98 737	15
46	9.37 652	51	9.38 918	54	0.61 082	9.98 734	14
47	9.37 703	52	9.38 972	55	0.61 028	9.98 731	13
48	9.37 755	51	9.39 027	55	0.60 973	9.98 728	12
49	9.37 806	52	9.39 082	54	0.60 918	9.98 725	11
50	9.37 858	51	9.39 136	54	0.60 864	9.98 722	10
51	9.37 909	51	9.39 190	55	0.60 810	9.98 719	9
52	9.37 960	51	9.39 245	54	0.60 755	9.98 715	8
53	9.38 011	51	9.39 299	54	0.60 701	9.98 712	7
54	9.38 062	51	9.39 353	54	0.60 647	9.98 709	6
55	9.38 113	51	9.39 407	54	0.60 593	9.98 706	5
56	9.38 164	51	9.39 461	54	0.60 539	9.98 703	4
57	9.38 215	51	9.39 515	54	0.60 485	9.98 700	3
58	9.38 266	51	9.39 569	54	0.60 431	9.98 697	2
59	9.38 317	51	9.39 623	54	0.60 377	9.98 694	1
60	9.38 368		9.39 677		0.60 323	9.98 690	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	58	57	56	55	54	53	52	51	4	3	2
.1	5.8	5.7	5.6	5.5	5.4	5.3	5.2	5.1	0.4	0.3	0.2
.2	11.6	11.4	11.2	11.0	10.8	10.6	10.4	10.2	0.8	0.6	0.4
.3	17.4	17.1	16.8	16.5	16.2	15.9	15.6	15.3	1.2	0.9	0.6
.4	23.2	22.8	22.4	22.0	21.6	21.2	20.8	20.4	1.6	1.2	0.8
.5	29.0	28.5	28.0	27.5	27.0	26.5	26.0	25.5	2.0	1.5	1.0
.6	34.8	34.2	33.6	33.0	32.4	31.8	31.2	30.6	2.4	1.8	1.2
.7	40.6	39.9	39.2	38.5	37.8	37.1	36.4	35.7	2.8	2.1	1.4
.8	46.4	45.6	44.8	44.0	43.2	42.4	41.6	40.8	3.2	2.4	1.6
.9	52.2	51.3	50.4	49.5	48.6	47.7	46.8	45.9	3.6	2.7	1.8

Prop. Pts.

76°

14°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.38 368	50	9.39 677	54	0.60 323	9.98 690	3	**60**
1	9.38 418	51	9.39 731	54	0.60 269	9.98 687	3	59
2	9.38 469	50	9.39 785̄	53	0.60 215	9.98 684	3	58
3	9.38 519	51	9.39 838	54	0.60 162	9.98 681	3	57
4	9.38 570	50	9.39 892	53	0.60 108	9.98 678	3	56
5	9.38 620	50	9.39 945	54	0.60 055̄	9.98 675̄	4	55
6	9.38 670	51	9.39 999	53	0.60 001	9.98 671	3	54
7	9.38 721	50	9.40 052	54	0.59 948	9.98 668	3	53
8	9.38 771	50	9.40 106	53	0.59 894	9.98 665	3	52
9	9.38 821	50	9.40 159	53	0.59 841	9.98 662	3	51
10	9.38 871	50	9.40 212	54	0.59 788	9.98 659	3	**50**
11	9.38 921	50	9.40 266	53	0.59 734	9.98 656	4	49
12	9.38 971	50	9.40 319	53	0.59 681	9.98 652	3	48
13	9.39 021	50	9.40 372	53	0.59 628	9.98 649	3	47
14	9.39 071	50	9.40 425̄	53	0.59 575	9.98 646	3	46
15	9.39 121	49	9.40 478	53	0.59 522	9.98 643	3	45
16	9.39 170	50	9.40 531	53	0.59 469	9.98 640	4	44
17	9.39 220	50	9.40 584	52	0.59 416	9.98 636	3	43
18	9.39 270	49	9.40 636	53	0.59 364	9.98 633	3	42
19	9.39 319	50	9.40 689	53	0.59 311	9.98 630	3	41
20	9.39 369	49	9.40 742	53	0.59 258	9.98 627	4	**40**
21	9.39 418	49	9.40 795̄	52	0.59 205	9.98 623	3	39
22	9.39 467	50	9.40 847	53	0.59 153	9.98 620	3	38
23	9.39 517	49	9.40 900	52	0.59 100	9.98 617	3	37
24	9.39 566	49	9.40 952	53	0.59 048	9.98 614	4	36
25	9.39 615̄	49	9.41 005̄	52	0.58 995	9.98 610	3	35
26	9.39 664	49	9.41 057	52	0.58 943	9.98 607	3	34
27	9.39 713	49	9.41 109	52	0.58 891	9.98 604	3	33
28	9.39 762	49	9.41 161	53	0.58 839	9.98 601	4	32
29	9.39 811	49	9.41 214	52	0.58 786	9.98 597	3	31
30	9.39 860	49	9.41 266	52	0.58 734	9.98 594	3	**30**
31	9.39 909	49	9.41 318	52	0.58 682	9.98 591	3	29
32	9.39 958	48	9.41 370	52	0.58 630	9.98 588	4	28
33	9.40 006	49	9.41 422	52	0.58 578	9.98 584	3	27
34	9.40 055̄	48	9.41 474	52	0.58 526	9.98 581	3	26
35	9.40 103	49	9.41 526	52	0.58 474	9.98 578	4	25
36	9.40 152	48	9.41 578	51	0.58 422	9.98 574	3	24
37	9.40 200	49.	9.41 629	52	0.58 371	9.98 571	3	23
38	9.40 249	48	9.41 681	52	0.58 319	9.98 568	3	22
39	9.40 297	49	9.41 733	51	0.58 267	9.98 565̄	4	21
40	9.40 346	48	9.41 784	52	0.58 216	9.98 561	3	**20**
41	9.40 394	48	9.41 836	51	0.58 164	9.98 558	3	19
42	9.40 442	48	9.41 887	52	0.58 113	9.98 555̄	4	18
43	9.40 490	48	9.41 939	51	0.58 061	9.98 551	3	17
44	9.40 538	48	9.41 990	51	0.58 010	9.98 548	3	16
45	9.40 586	48	9.42 041	52	0.57 959	9.98 545̄	4	15
46	9.40 634	48	9.42 093	51	0.57 907	9.98 541	3	14
47	9.40 682	48	9.42 144	51	0.57 856	9.98 538	3	13
48	9.40 730	48	9.42 195	51	0.57 805̄	9.98 535̄	4	12
49	9.40 778	47	9.42 246	51	0.57 754	9.98 531	3	11
50	9.40 825	48	9.42 297	51	0.57 703	9.98 528	3	**10**
51	9.40 873	48	9.42 348	51	0.57 652	9.98 525̄	4	9
52	9.40 921	47	9.42 399	51	0.57 601	9.98 521	3	8
53	9.40 968	48	9.42 450	51	0.57 55̄0	9.98 518	3	7
54	9.41 016	47	9.42 501	51	0.57 499	9.98 515̄	4	6
55	9.41 063	48	9.42 552	51	0.57 448	9.98 511	3	5
56	9.41 111	47	9.42 603	50	0.57 397	9.98 508	3	4
57	9.41 158	47	9.42 653	51	0.57 347	9.98 505̄	4	3
58	9.41 205	47	9.42 704	51	0.57 296	9.98 501	3	2
59	9.41 252	48	9.42 755̄	50	0.57 245	9.98 498	4	1
60	9.41 300		9.42 805		0.57 195̄	9.98 494		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	54	53
.1	5.4	5.3
.2	10.8	10.6
.3	16.2	15.9
.4	21.6	21.2
.5	27.0	26.5
.6	32.4	31.8
.7	37.8	37.1
.8	43.2	42.4
.9	48.6	47.7

	52	51
.1	5.2	5.1
.2	10.4	10.2
.3	15.6	15.3
.4	20.8	20.4
.5	26.0	25.5
.6	31.2	30.6
.7	36.4	35.7
.8	41.6	40.8
.9	46.8	45.9

	50	49
.1	5.0	4.9
.2	10.0	9.8
.3	15.0	14.7
.4	20.0	19.6
.5	25.0	24.5
.6	30.0	29.4
.7	35.0	34.3
.8	40.0	39.2
.9	45.0	44.1

	48	47
.1	4.8	4.7
.2	9.6	9.4
.3	14.4	14.1
.4	19.2	18.8
.5	24.0	23.5
.6	28.8	28.2
.7	33.6	32.9
.8	38.4	37.6
.9	43.2	42.3

	4	3
.1	0.4	0.3
.2	0.8	0.6
.3	1.2	0.9
.4	1.6	1.2
.5	2.0	1.5
.6	2.4	1.8
.7	2.8	2.1
.8	3.2	2.4
.9	3.6	2.7

Prop. Pts.

75°

15°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.41 300	47	9.42 805	51	0.57 19$\bar{5}$	9.98 494	3	60
1	9.41 347	47	9.42 856	50	0.57 144	9.98 491	3	59
2	9.41 394	47	9.42 906	51	0.57 094	9.98 488	4	58
3	9.41 441	47	9.42 957	50	0.57 043	9.98 484	3	57
4	9.41 488	47	9.43 007	50	0.56 993	9.98 481	4	56
5	9.41 53$\bar{5}$	47	9.43 057	51	0.56 943	9.98 477	3	55
6	9.41 582	46	9.43 108	50	0.56 892	9.98 474	3	54
7	9.41 628	47	9.43 158	50	0.56 842	9.98 471	4	53
8	9.41 675	47	9.43 208	50	0.56 792	9.98 467	3	52
9	9.41 722	46	9.43 258	50	0.56 742	9.98 464	4	51
10	9.41 768	47	9.43 308	50	0.56 692	9.98 460	3	50
11	9.41 81$\bar{5}$	46	9.43 358	50	0.56 642	9.98 457	4	49
12	9.41 861	47	9.43 408	50	0.56 592	9.98 453	3	48
13	9.41 908	46	9.43 458	50	0.56 542	9.98 450	3	47
14	9.41 954	47	9.43 508	50	0.56 492	9.98 447	4	46
15	9.42 001	46	9.43 558	49	0.56 442	9.98 443	3	45
16	9.42 047	46	9.43 607	50	0.56 393	9.98 440	4	44
17	9.42 093	47	9.43 657	50	0.56 343	9.98 436	3	43
18	9.42 140	46	9.43 707	49	0.56 293	9.98 433	4	42
19	9.42 186	46	9.43 756	50	0.56 244	9.98 429	3	41
20	9.42 232	46	9.43 806	49	0.56 194	9.98 426	4	40
21	9.42 278	46	9.43 855	50	0.56 14$\bar{5}$	9.98 422	3	39
22	9.42 324	46	9.43 90$\bar{5}$	49	0.56 095	9.98 419	4	38
23	9.42 370	46	9.43 954	50	0.56 046	9.98 415	3	37
24	9.42 416	45	9.44 004	49	0.55 996	9.98 412	3	36
25	9.42 461	46	9.44 053	49	0.55 947	9.98 409	4	35
26	9.42 507	46	9.44 102	49	0.55 898	9.98 405	3	34
27	9.42 553	46	9.44 151	50	0.55 849	9.98 402	4	33
28	9.42 599	45	9.44 201	49	0.55 799	9.98 398	3	32
29	9.42 644	46	9.44 2$\bar{5}$0	49	0.55 750	9.98 39$\bar{5}$	4	31
30	9.42 690	45	9.44 299	49	0.55 701	9.98 391	3	30
31	9.42 735	46	9.44 348	49	0.55 652	9.98 388	4	29
32	9.42 781	45	9.44 397	49	0.55 603	9.98 384	3	28
33	9.42 826	46	9.44 446	49	0.55 554	9.98 381	4	27
34	9.42 872	45	9.44 49$\bar{5}$	49	0.55 505	9.98 377	4	26
35	9.42 917	45	9.44 544	48	0.55 456	9.98 373	3	25
36	9.42 962	46	9.44 592	49	0.55 408	9.98 370	4	24
37	9.43 008	45	9.44 641	49	0.55 359	9.98 366	3	23
38	9.43 053	45	9.44 690	48	0.55 310	9.98 363	4	22
39	9.43 098	45	9.44 738	49	0.55 262	9.98 359	3	21
40	9.43 143	45	9.44 787	49	0.55 213	9.98 356	4	20
41	9.43 188	45	9.44 836	48	0.55 164	9.98 352	3	19
42	9.43 233	45	9.44 884	49	0.55 116	9.98 349	4	18
43	9.43 278	45	9.44 933	48	0.55 067	9.98 345	3	17
44	9.43 323	44	9.44 981	48	0.55 019	9.98 342	4	16
45	9.43 367	45	9.45 029	49	0.54 971	9.98 338	4	15
46	9.43 412	45	9.45 078	48	0.54 922	9.98 334	3	14
47	9.43 457	45	9.45 126	48	0.54 874	9.98 331	4	13
48	9.43 502	44	9.45 174	48	0.54 826	9.98 327	3	12
49	9.43 546	45	9.45 222	49	0.54 778	9.98 324	4	11
50	9.43 591	44	9.45 271	48	0.54 729	9.98 320	3	10
51	9.43 635	45	9.45 319	48	0.54 681	9.98 317	4	9
52	9.43 680	44	9.45 367	48	0.54 633	9.98 313	4	8
53	9.43 724	45	9.45 41$\bar{5}$	48	0.54 585	9.98 309	3	7
54	9.43 769	44	9.45 463	48	0.54 537	9.98 306	4	6
55	9.43 813	44	9.45 511	48	0.54 489	9.98 302	3	5
56	9.43 857	44	9.45 559	47	0.54 441	9.98 299	4	4
57	9.43 901	45	9.45 606	48	0.54 394	9.98 295	4	3
58	9.43 946	44	9.45 654	48	0.54 346	9.98 291	3	2
59	9.43 990	44	9.45 702	48	0.54 298	9.98 288	4	1
60	9.44 034		9.45 7$\bar{5}$0		0.54 250	9.98 284		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	51	50
.1	5.1	5.0
.2	10.2	10.0
.3	15.3	15.0
.4	20.4	20.0
.5	25.5	25.0
.6	30.6	30.0
.7	35.7	35.0
.8	40.8	40.0
.9	45.9	45.0

	49	48
.1	4.9	4.8
.2	9.8	9.6
.3	14.7	14.4
.4	19.6	19.2
.5	24.5	24.0
.6	29.4	28.8
.7	34.3	33.6
.8	39.2	38.4
.9	44.1	43.2

	47	46
.1	4.7	4.6
.2	9.4	9.2
.3	14.1	13.8
.4	18.8	18.4
.5	23.5	23.0
.6	28.2	27.6
.7	32.9	32.2
.8	37.6	36.8
.9	42.3	41.4

	45	44
.1	4.5	4.4
.2	9.0	8.8
.3	13.5	13.2
.4	18.0	17.6
.5	22.5	22.0
.6	27.0	26.4
.7	31.5	30.8
.8	36.0	35.2
.9	40.5	39.6

	4	3
.1	0.4	0.3
.2	0.8	0.6
.3	1.2	0.9
.4	1.6	1.2
.5	2.0	1.5
.6	2.4	1.8
.7	2.8	2.1
.8	3.2	2.4
.9	3.6	2.7

Prop. Pts.

74°

16°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.44 034	44	9.45 750	47	0.54 250	9.98 284	3	60
1	9.44 078	44	9.45 797	48	0.54 203	9.98 281	4	59
2	9.44 122	44	9.45 845	47	0.54 155	9.98 277	4	58
3	9.44 166	44	9.45 892	48	0.54 108	9.98 273	3	57
4	9.44 210	43	9.45 940	47	0.54 060	9.98 270	4	56
5	9.44 253	44	9.45 987	48	0.54 013	9.98 266	4	55
6	9.44 297	44	9.46 035	47	0.53 965	9.98 262	3	54
7	9.44 341	44	9.46 082	48	0.53 918	9.98 259	4	53
8	9.44 385	43	9.46 130	47	0.53 870	9.98 255	4	52
9	9.44 428	44	9.46 177	47	0.53 823	9.98 251	3	51
10	9.44·472	44	9.46 224	47	0.53 776	9.98 248	4	50
11	9.44 516	43	9.46 271	48	0.53 729	9.98 244	4	49
12	9.44 559	43	9.46 319	47	0.53 681	9.98 240	3	48
13	9.44 602	44	9.46 366	47	0.53 634	9.98 237	4	47
14	9.44 646	43	9.46 413	47	0.53 587	9.98 233	4	46
15	9.44 689	44	9.46 460	47	0.53 540	9.98 229	3	45
16	9.44 733	43	9.46 507	47	0.53 493	9.98 226	4	44
17	9.44 776	43	9.46 554	47	0.53 446	9.98 222	4	43
18	9.44 819	43	9.46 601	47	0.53 399	9.98 218	3	42
19	9.44 862	43	9.46 648	46	0.53 352	9.98 215	4	41
20	9.44 905	43	9.46 694	47	0.53 306	9.98 211	4	40
21	9.44 948	44	9.46 741	47	0.53 259	9.98 207	3	39
22	9.44 992	43	9.46 788	47	0.53 212	9.98 204	4	38
23	9.45 035	42	9.46 835	46	0.53 165	9.98 200	4	37
24	9.45 077	43	9.46 881	47	0.53 119	9.98 196	4	36
25	9.45 120	43	9.46 928	47	0.53 072	9.98 192	3	35
26	9.45 163	43	9.46 975	46	0.53 025	9.98 189	4	34
27	9.45 206	43	9.47 021	47	0.52 979	9.98 185	4	33
28	9.45 249	43	9.47 068	46	0.52 932	9.98 181	4	32
29	9.45 292	42	9.47 114	46	0.52 886	9.98 177	3	31
30	9.45 334	43	9.47 160	47	0.52 840	9.98 174	4	30
31	9.45 377	42	9.47 207	46	0.52 793	9.98 170	4	29
32	9.45 419	43	9.47 253	46	0.52 747	9.98 166	4	28
33	9.45 462	42	9.47 299	47	0.52 701	9.98 162	3	27
34	9.45 504	43	9.47 346	46	0.52 654	9.98 159	4	26
35	9.45 547	42	9.47 392	46	0.52 608	9.98 155	4	25
36	9.45 589	43	9.47 438	46	0.52 562	9.98 151	4	24
37	9.45 632	42	9.47 484	46	0.52 516	9.98 147	3	23
38	9.45 674	42	9.47 530	46	0.52 470	9.98 144	4	22
39	9.45 716	42	9.47 576	46	0.52 424	9.98 140	4	21
40	9.45 758	43	9.47 622	46	0.52 378	9.98 136	4	20
41	9.45 801	42	9.47 668	46	0.52 332	9.98 132	3	19
42	9.45 843	42	9.47 714	46	0.52 286	9.98 129	4	18
43	9.45 885	42	9.47 760	46	0.52 240	9.98 125	4	17
44	9.45 927	42	9.47 806	46	0.52 194	9.98 121	4	16
45	9.45 969	42	9.47 852	45	0.52 148	9.98 117	4	15
46	9.46 011	42	9.47 897	46	0.52 103	9.98 113	3	14
47	9.46 053	42	9.47 943	46	0.52 057	9.98 110	4	13
48	9.46 095	41	9.47 989	46	0.52 011	9.98 106	4	12
49	9.46 136	42	9.48 035	45	0.51 965	9.98 102	4	11
50	9.46 178	42	9.48 080	46	0.51 920	9.98 098	4	10
51	9.46 220	42	9.48 126	45	0.51 874	9.98 094	4	9
52	9.46 262	41	9.48 171	46	0.51 829	9.98 090	3	8
53	9.46 303	42	9.48 217	45	0.51 783	9.98 087	4	7
54	9.46 345	41	9.48 262	45	0.51 738	9.98 083	4	6
55	9.46 386	42	9.48 307	46	0.51 693	9.98 079	4	5
56	9.46 428	41	9.48 353	45	0.51 647	9.98 075	4	4
57	9.46 469	42	9.48 398	45	0.51 602	9.98 071	4	3
58	9.46 511	41	9.48 443	46	0.51 557	9.98 067	4	2
59	9.46 552	42	9.48 489	45	0.51 511	9.98 063	3	1
60	9.46 594		9.48 534		0.51 466	9.98 060		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

73°

Prop. Pts.

	48	47
.1	4.8	4.7
.2	9.6	9.4
.3	14.4	14.1
.4	19.2	18.8
.5	24.0	23.5
.6	28.8	28.2
.7	33.6	32.9
.8	38.4	37.6
.9	43.2	42.3

	46	45
.1	4.6	4.5
.2	9.2	9.0
.3	13.8	13.5
.4	18.4	18.0
.5	23.0	22.5
.6	27.6	27.0
.7	32.2	31.5
.8	36.8	36.0
.9	41.4	40.5

	44	43
.1	4.4	4.3
.2	8.8	8.6
.3	13.2	12.9
.4	17.6	17.2
.5	22.0	21.5
.6	26.4	25.8
.7	30.8	30.1
.8	35.2	34.4
.9	39.6	38.7

	42	41
.1	4.2	4.1
.2	8.4	8.2
.3	12.6	12.3
.4	16.8	16.4
.5	21.0	20.5
.6	25.2	24.6
.7	29.4	28.7
.8	33.6	32.8
.9	37.8	36.9

	4	3
.1	0.4	0.3
.2	0.8	0.6
.3	1.2	0.9
.4	1.6	1.2
.5	2.0	1.5
.6	2.4	1.8
.7	2.8	2.1
.8	3.2	2.4
.9	3.6	2.7

Prop. Pts.

17°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.46 594	41	9.48 534	45	0.51 466	9.98 060	4	60
1	9.46 635	41	9.48 579	45	0.51 421	9.98 056	4	59
2	9.46 676	41	9.48 624	45	0.51 376	9.98 052	4	58
3	9.46 717	41	9.48 669	45	0.51 331	9.98 048	4	57
4	9.46 758	42	9.48 714	45	0.51 286	9.98 044	4	56
5	9.46 800	41	9.48 759	45	0.51 241	9.98 040	4	55
6	9.46 841	41	9.48 804	45	0.51 196	9.98 036	4	54
7	9.46 882	41	9.48 849	45	0.51 151	9.98 032	3	53
8	9.46 923	41	9.48 894	45	0.51 106	9.98 029	4	52
9	9.46 964	41	9.48 939	45	0.51 061	9.98 025	4	51
10	9.47 005	40	9.48 984	45	0.51 016	9.98 021	4	50
11	9.47 045	41	9.49 029	44	0.50 971	9.98 017	4	49
12	9.47 086	41	9.49 073	45	0.50 927	9.98 013	4	48
13	9.47 127	41	9.49 118	45	0.50 882	9.98 009	4	47
14	9.47 168	41	9.49 163	44	0.50 837	9.98 005	4	46
15	9.47 209	40	9.49 207	45	0.50 793	9.98 001	4	45
16	9.47 249	41	9.49 252	44	0.50 748	9.97 997	4	44
17	9.47 290	40	9.49 296	45	0.50 704	9.97 993	4	43
18	9.47 330	41	9.49 341	44	0.50 659	9.97 989	3	42
19	9.47 371	40	9.49 385	45	0.50 615	9.97 986	4	41
20	9.47 411	41	9.49 430	44	0.50 570	9.97 982	4	40
21	9.47 452	40	9.49 474	45	0.50 526	9.97 978	4	39
22	9.47 492	41	9.49 519	44	0.50 481	9.97 974	4	38
23	9.47 533	40	9.49 563	44	0.50 437	9.97 970	4	37
24	9.47 573	40	9.49 607	45	0.50 393	9.97 966	4	36
25	9.47 613	41	9.49 652	44	0.50 348	9.97 962	4	35
26	9.47 654	40	9.49 696	44	0.50 304	9.97 958	4	34
27	9.47 694	40	9.49 740	44	0.50 260	9.97 954	4	33
28	9.47 734	40	9.49 784	44	0.50 216	9.97 950	4	32
29	9.47 774	40	9.49 828	44	0.50 172	9.97 946	4	31
30	9.47 814	40	9.49 872	44	0.50 128	9.97 942	4	30
31	9.47 854	40	9.49 916	44	0.50 084	9.97 938	4	29
32	9.47 894	40	9.49 960	44	0.50 040	9.97 934	4	28
33	9.47 934	40	9.50 004	44	0.49 996	9.97 930	4	27
34	9.47 974	40	9.50 048	44	0.49 952	9.97 926	4	26
35	9.48 014	40	9.50 092	44	0.49 908	9.97 922	4	25
36	9.48 054	40	9.50 136	44	0.49 864	9.97 918	4	24
37	9.48 094	39	9.50 180	43	0.49 820	9.97 914	4	23
38	9.48 133	40	9.50 223	44	0.49 777	9.97 910	4	22
39	9.48 173	40	9.50 267	44	0.49 733	9.97 906	4	21
40	9.48 213	39	9.50 311	44	0.49 689	9.97 902	4	20
41	9.48 252	40	9.50 355	43	0.49 645	9.97 898	4	19
42	9.48 292	40	9.50 398	44	0.49 602	9.97 894	4	18
43	9.48 332	39	9.50 442	43	0.49 558	9.97 890	4	17
44	9.48 371	40	9.50 485	44	0.49 515	9.97 886	4	16
45	9.48 411	39	9.50 529	43	0.49 471	9.97 882	4	15
46	9.48 450	40	9.50 572	44	0.49 428	9.97 878	4	14
47	9.48 490	39	9.50 616	43	0.49 384	9.97 874	4	13
48	9.48 529	39	9.50 659	44	0.49 341	9.97 870	4	12
49	9.48 568	39	9.50 703	43	0.49 297	9.97 866	5	11
50	9.48 607	40	9.50 746	43	0.49 254	9.97 861	4	10
51	9.48 647	39	9.50 789	44	0.49 211	9.97 857	4	9
52	9.48 686	39	9.50 833	43	0.49 167	9.97 853	4	8
53	9.48 725	39	9.50 876	43	0.49 124	9.97 849	4	7
54	9.48 764	39	9.50 919	43	0.49 081	9.97 845	4	6
55	9.48 803	39	9.50 962	43	0.49 038	9.97 841	4	5
56	9.48 842	39	9.51 005	43	0.48 995	9.97 837	4	4
57	9.48 881	39	9.51 048	44	0.48 952	9.97 833	4	3
58	9.48 920	39	9.51 092	43	0.48 908	9.97 829	4	2
59	9.48 959	39	9.51 135	43	0.48 865	9.97 825	4	1
60	9.48 998		9.51 178		0.48 822	9.97 821		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	45	44
.1	4.5	4.4
.2	9.0	8.8
.3	13.5	13.2
.4	18.0	17.6
.5	22.5	22.0
.6	27.0	26.4
.7	31.5	30.8
.8	36.0	35.2
.9	40.5	39.6

	43	42
.1	4.3	4.2
.2	8.6	8.4
.3	12.9	12.6
.4	17.2	16.8
.5	21.5	21.0
.6	25.8	25.2
.7	30.1	29.4
.8	34.4	33.6
.9	38.7	37.8

	41	40
.1	4.1	4.0
.2	8.2	8.0
.3	12.3	12.0
.4	16.4	16.0
.5	20.5	20.0
.6	24.6	24.0
.7	28.7	28.0
.8	32.8	32.0
.9	36.9	36.0

	39	5
.1	3.9	0.5
.2	7.8	1.0
.3	11.7	1.5
.4	15.6	2.0
.5	19.5	2.5
.6	23.4	3.0
.7	27.3	3.5
.8	31.2	4.0
.9	35.1	4.5

	4	3
.1	0.4	0.3
.2	0.8	0.6
.3	1.2	0.9
.4	1.6	1.2
.5	2.0	1.5
.6	2.4	1.8
.7	2.8	2.1
.8	3.2	2.4
.9	3.6	2.7

Prop. Pts.

72°

18°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.48 998	39	9.51 178	43	0.48 822	9.97 821	4	60
1	9.49 037	39	9.51 221	43	0.48 779	9.97 817	5	59
2	9.49 076	39	9.51 264	42	0.48 736	9.97 812	4	58
3	9.49 115	38	9.51 306	43	0.48 694	9.97 808	4	57
4	9.49 153	39	9.51 349	43	0.48 651	9.97 804	4	56
5	9.49 192	39	9.51 392	43	0.48 608	9.97 800	4	55
6	9.49 231	38	9.51 435	43	0.48 565	9.97 796	4	54
7	9.49 269	39	9.51 478	42	0.48 522	9.97 792	4	53
8	9.49 308	39	9.51 520	43	0.48 480	9.97 788	4	52
9	9.49 347	38	9.51 563	43	0.48 437	9.97 784	5	51
10	9.49 385	39	9.51 606	42	0.48 394	9.97 779	4	50
11	9.49 424	38	9.51 648	43	0.48 352	9.97 775	4	49
12	9.49 462	38	9.51 691	43	0.48 309	9.97 771	4	48
13	9.49 500	39	9.51 734	42	0.48 266	9.97 767	4	47
14	9.49 539	38	9.51 776	43	0.48 224	9.97 763	4	46
15	9.49 577	38	9.51 819	42	0.48 181	9.97 759	5	45
16	9.49 615	39	9.51 861	42	0.48 139	9.97 754	4	44
17	9.49 654	38	9.51 903	43	0.48 097	9.97 750	4	43
18	9.49 692	38	9.51 946	42	0.48 054	9.97 746	4	42
19	9.49 730	38	9.51 988	43	0.48 012	9.97 742	4	41
20	9.49 768	38	9.52 031	42	0.47 969	9.97 738	4	40
21	9.49 806	38	9.52 073	42	0.47 927	9.97 734	5	39
22	9.49 844	38	9.52 115	42	0.47 885	9.97 729	4	38
23	9.49 882	38	9.52 157	43	0.47 843	9.97 725	4	37
24	9.49 920	38	9.52 200	42	0.47 800	9.97 721	4	36
25	9.49 958	38	9.52 242	42	0.47 758	9.97 717	4	35
26	9.49 996	38	9.52 284	42	0.47 716	9.97 713	5	34
27	9.50 034	38	9.52 326	42	0.47 674	9.97 708	4	33
28	9.50 072	38	9.52 368	42	0.47 632	9.97 704	4	32
29	9.50 110	38	9.52 410	42	0.47 590	9.97 700	4	31
30	9.50 148	37	9.52 452	42	0.47 548	9.97 696	5	30
31	9.50 185	38	9.52 494	42	0.47 506	9.97 691	4	29
32	9.50 223	38	9.52 536	42	0.47 464	9.97 687	4	28
33	9.50 261	37	9.52 578	42	0.47 422	9.97 683	4	27
34	9.50 298	38	9.52 620	41	0.47 380	9.97 679	5	26
35	9.50 336	38	9.52 661	42	0.47 339	9.97 674	4	25
36	9.50 374	37	9.52 703	42	0.47 297	9.97 670	4	24
37	9.50 411	38	9.52 745	42	0.47 255	9.97 666	4	23
38	9.50 449	37	9.52 787	42	0.47 213	9.97 662	5	22
39	9.50 486	37	9.52 829	41	0.47 171	9.97 657	4	21
40	9.50 523	38	9.52 870	42	0.47 130	9.97 653	4	20
41	9.50 561	37	9.52 912	41	0.47 088	9.97 649	4	19
42	9.50 598	37	9.52 953	42	0.47 047	9.97 645	5	18
43	9.50 635	38	9.52 995	42	0.47 005	9.97 640	4	17
44	9.50 673	37	9.53 037	41	0.46 963	9.97 636	4	16
45	9.50 710	37	9.53 078	42	0.46 922	9.97 632	4	15
46	9.50 747	37	9.53 120	41	0.46 880	9.97 628	5	14
47	9.50 784	37	9.53 161	41	0.46 839	9.97 623	4	13
48	9.50 821	37	9.53 202	42	0.46 798	9.97 619	4	12
49	9.50 858	38	9.53 244	41	0.46 756	9.97 615	5	11
50	9.50 896	37	9.53 285	42	0.46 715	9.97 610	4	10
51	9.50 933	37	9.53 327	41	0.46 673	9.97 606	4	9
52	9.50 970	37	9.53 368	41	0.46 632	9.97 602	5	8
53	9.51 007	36	9.53 409	41	0.46 591	9.97 597	4	7
54	9.51 043	37	9.53 450	42	0.46 550	9.97 593	4	6
55	9.51 080	37	9.53 492	41	0.46 508	9.97 589	5	5
56	9.51 117	37	9.53 533	41	0.46 467	9.97 584	4	4
57	9.51 154	37	9.53 574	41	0.46 426	9.97 580	4	3
58	9.51 191	36	9.53 615	41	0.46 385	9.97 576	5	2
59	9.51 227	37	9.53 656	41	0.46 344	9.97 571	4	1
60	9.51 264		9.53 697		0.46 303	9.97 567		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	43	42
.1	4.3	4.2
.2	8.6	8.4
.3	12.9	12.6
.4	17.2	16.8
.5	21.5	21.0
.6	25.8	25.2
.7	30.1	29.4
.8	34.4	33.6
.9	38.7	37.8

	41
.1	4.1
.2	8.2
.3	12.3
.4	16.4
.5	20.5
.6	24.6
.7	28.7
.8	32.8
.9	36.9

	39	38
.1	3.9	3.8
.2	7.8	7.6
.3	11.7	11.4
.4	15.6	15.2
.5	19.5	19.0
.6	23.4	22.8
.7	27.3	26.6
.8	31.2	30.4
.9	35.1	34.2

	37	36
.1	3.7	3.6
.2	7.4	7.2
.3	11.1	10.8
.4	14.8	14.4
.5	18.5	18.0
.6	22.2	21.6
.7	25.9	25.2
.8	29.6	28.8
.9	33.3	32.4

	5	4
.1	0.5	0.4
.2	1.0	0.8
.3	1.5	1.2
.4	2.0	1.6
.5	2.5	2.0
.6	3.0	2.4
.7	3.5	2.8
.8	4.0	3.2
.9	4.5	3.6

Prop. Pts.

71°

19°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.51 264	37	9.53 697	41	0.46 303	9.97 567	4	**60**
1	9.51 301	37	9.53 738	41	0.46 262	9.97 563	5	59
2	9.51 338	36	9.53 779	41	0.46 221	9.97 558	4	58
3	9.51 374	37	9.53 820	41	0.46 180	9.97 554	4	57
4	9.51 411	36	9.53 861	41	0.46 139	9.97 550	5	56
5	9.51 447	37	9.53 902	41	0.46 098	9.97 545	4	55
6	9.51 484	36	9.53 943	41	0.46 057	9.97 541	5	54
7	9.51 520	37	9.53 984	41	0.46 016	9.97 536	4	53
8	9.51 557	36	9.54 025	40	0.45 975	9.97 532	4	52
9	9.51 593	36	9.54 065	41	0.45 935	9.97 528	5	51
10	9.51 629	37	9.54 106	41	0.45 894	9.97 523	4	**50**
11	9.51 666	36	9.54 147	40	0.45 853	9.97 519	4	49
12	9.51 702	36	9.54 187	41	0.45 813	9.97 515	5	48
13	9.51 738	36	9.54 228	41	0.45 772	9.97 510	4	47
14	9.51 774	37	9.54 269	40	0.45 731	9.97 506	5	46
15	9.51 811	36	9.54 309	41	0.45 691	9.97 501	4	45
16	9.51 847	36	9.54 350	40	0.45 650	9.97 497	5	44
17	9.51 883	36	9.54 390	41	0.45 610	9.97 492	4	43
18	9.51 919	36	9.54 431	40	0.45 569	9.97 488	4	42
19	9.51 955	36	9.54 471	41	0.45 529	9.97 484	5	41
20	9.51 991	36	9.54 512	40	0.45 488	9.97 479	4	**40**
21	9.52 027	36	9.54 552	41	0.45 448	9.97 475	5	39
22	9.52 063	36	9.54 593	40	0.45 407	9.97 470	4	38
23	9.52 099	36	9.54 633	40	0.45 367	9.97 466	5	37
24	9.52 135	36	9.54 673	41	0.45 327	9.97 461	4	36
25	9.52 171	36	9.54 714	40	0.45 286	9.97 457	4	35
26	9.52 207	35	9.54 754	40	0.45 246	9.97 453	5	34
27	9.52 242	36	9.54 794	41	0.45 206	9.97 448	4	33
28	9.52 278	36	9.54 835	40	0.45 165	9.97 444	5	32
29	9.52 314	36	9.54 875	40	0.45 125	9.97 439	4	31
30	9.52 350	35	9.54 915	40	0.45 085	9.97 435	5	**30**
31	9.52 385	36	9.54 955	40	0.45 045	9.97 430	4	29
32	9.52 421	35	9.54 995	40	0.45 005	9.97 426	5	28
33	9.52 456	36	9.55 035	40	0.44 965	9.97 421	4	27
34	9.52 492	35	9.55 075	40	0.44 925	9.97 417	5	26
35	9.52 527	36	9.55 115	40	0.44 885	9.97 412	4	25
36	9.52 563	35	9.55 155	40	0.44 845	9.97 408	5	24
37	9.52 598	36	9.55 195	40	0.44 805	9.97 403	4	23
38	9.52 634	35	9.55 235	40	0.44 765	9.97 399	5	22
39	9.52 669	36	9.55 275	40	0.44 725	9.97 394	4	21
40	9.52 705	35	9.55 315	40	0.44 685	9.97 390	5	**20**
41	9.52 740	35	9.55 355	40	0.44 645	9.97 385	4	19
42	9.52 775	36	9.55 395	39	0.44 605	9.97 381	5	18
43	9.52 811	35	9.55 434	40	0.44 566	9.97 376	4	17
44	9.52 846	35	9.55 474	40	0.44 526	9.97 372	5	16
45	9.52 881	35	9.55 514	40	0.44 486	9.97 367	4	15
46	9.52 916	35	9.55 554	39	0.44 446	9.97 363	5	14
47	9.52 951	35	9.55 593	40	0.44 407	9.97 358	5	13
48	9.52 986	35	9.55 633	40	0.44 367	9.97 353	4	12
49	9.53 021	35	9.55 673	39	0.44 327	9.97 349	5	11
50	9.53 056	36	9.55 712	40	0.44 288	9.97 344	4	**10**
51	9.53 092	34	9.55 752	39	0.44 248	9.97 340	5	9
52	9.53 126	35	9.55 791	40	0.44 209	9.97 335	4	8
53	9.53 161	35	9.55 831	39	0.44 169	9.97 331	5	7
54	9.53 196	35	9.55 870	40	0.44 130	9.97 326	4	6
55	9.53 231	35	9.55 910	39	0.44 090	9.97 322	5	5
56	9.53 266	35	9.55 949	40	0.44 051	9.97 317	5	4
57	9.53 301	35	9.55 989	39	0.44 011	9.97 312	4	3
58	9.53 336	34	9.56 028	39	0.43 972	9.97 308	5	2
59	9.53 370	35	9.56 067	40	0.43 933	9.97 303	4	1
60	9.53 405		9.56 107		0.43 893	9.97 299		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	41	40
.1	4.1	4.0
.2	8.2	8.0
.3	12.3	12.0
.4	16.4	16.0
.5	20.5	20.0
.6	24.6	24.0
.7	28.7	28.0
.8	32.8	32.0
.9	36.9	36.0

	39
.1	3.9
.2	7.8
.3	11.7
.4	15.6
.5	19.5
.6	23.4
.7	27.3
.8	31.2
.9	35.1

	37	36
.1	3.7	3.6
.2	7.4	7.2
.3	11.1	10.8
.4	14.8	14.4
.5	18.5	18.0
.6	22.2	21.6
.7	25.9	25.2
.8	29.6	28.8
.9	33.3	32.4

	35	34
.1	3.5	3.4
.2	7.0	6.8
.3	10.5	10.2
.4	14.0	13.6
.5	17.5	17.0
.6	21.0	20.4
.7	24.5	23.8
.8	28.0	27.2
.9	31.5	30.6

	5	4
.1	0.5	0.4
.2	1.0	0.8
.3	1.5	1.2
.4	2.0	1.6
.5	2.5	2.0
.6	3.0	2.4
.7	3.5	2.8
.8	4.0	3.2
.9	4.5	3.6

Prop. Pts.

70°

20°

'	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.53 405	35	9.56 107	39	0.43 893	9.97 299	5	60
1	9.53 440	35	9.56 146	39	0.43 854	9.97 294	5	59
2	9.53 475	34	9.56 185	39	0.43 815	9.97 289	4	58
3	9.53 509	35	9.56 224	40	0.43 776	9.97 285	5	57
4	9.53 544	34	9.56 264	39	0.43 736	9.97 280	4	56
5	9.53 578	35	9.56 303	39	0.43 697	9.97 276	5	55
6	9.53 613	34	9.56 342	39	0.43 658	9.97 271	5	54
7	9.53 647	35	9.56 381	39	0.43 619	9.97 266	4	53
8	9.53 682	34	9.56 420	39	0.43 580	9.97 262	5	52
9	9.53 716	35	9.56 459	39	0.43 541	9.97 257	5	51
10	9.53 751	34	9.56 498	39	0.43 502	9.97 252	4	50
11	9.53 785	34	9.56 537	39	0.43 463	9.97 248	5	49
12	9.53 819	35	9.56 576	39	0.43 424	9.97 243	5	48
13	9.53 854	34	9.56 615	39	0.43 385	9.97 238	4	47
14	9.53 888	34	9.56 654	39	0.43 346	9.97 234	5	46
15	9.53 922	35	9.56 693	39	0.43 307	9.97 229	5	45
16	9.53 957	34	9.56 732	39	0.43 268	9.97 224	4	44
17	9.53 991	34	9.56 771	39	0.43 229	9.97 220	5	43
18	9.54 025	34	9.56 810	39	0.43 190	9.97 215	5	42
19	9.54 059	34	9.56 849	38	0.43 151	9.97 210	4	41
20	9.54 093	34	9.56 887	39	0.43 113	9.97 206	5	40
21	9.54 127	34	9.56 926	39	0.43 074	9.97 201	5	39
22	9.54 161	34	9.56 965	39	0.43 035	9.97 196	4	38
23	9.54 195	34	9.57 004	38	0.42 996	9.97 192	5	37
24	9.54 229	34	9.57 042	39	0.42 958	9.97 187	5	36
25	9.54 263	34	9.57 081	39	0.42 919	9.97 182	4	35
26	9.54 297	34	9.57 120	38	0.42 880	9.97 178	5	34
27	9.54 331	34	9.57 158	39	0.42 842	9.97 173	5	33
28	9.54 365	34	9.57 197	38	0.42 803	9.97 168	5	32
29	9.54 399	34	9.57 235	39	0.42 765	9.97 163	4	31
30	9.54 433	33	9.57 274	38	0.42 726	9.97 159	5	30
31	9.54 466	34	9.57 312	39	0.42 688	9.97 154	5	29
32	9.54 500	34	9.57 351	38	0.42 649	9.97 149	4	28
33	9.54 534	33	9.57 389	39	0.42 611	9.97 145	5	27
34	9.54 567	34	9.57 428	38	0.42 572	9.97 140	5	26
35	9.54 601	34	9.57 466	38	0.42 534	9.97 135	5	25
36	9.54 635	33	9.57 504	39	0.42 496	9.97 130	4	24
37	9.54 668	34	9.57 543	38	0.42 457	9.97 126	5	23
38	9.54 702	33	9.57 581	38	0.42 419	9.97 121	5	22
39	9.54 735	34	9.57 619	39	0.42 381	9.97 116	5	21
40	9.54 769	33	9.57 658	38	0.42 342	9.97 111	4	20
41	9.54 802	34	9.57 696	38	0.42 304	9.97 107	5	19
42	9.54 836	33	9.57 734	38	0.42 266	9.97 102	5	18
43	9.54 869	34	9.57 772	38	0.42 228	9.97 097	5	17
44	9.54 903	33	9.57 810	39	0.42 190	9.97 092	5	16
45	9.54 936	33	9.57 849	38	0.42 151	9.97 087	4	15
46	9.54 969	34	9.57 887	38	0.42 113	9.97 083	5	14
47	9.55 003	33	9.57 925	38	0.42 075	9.97 078	5	13
48	9.55 036	33	9.57 963	38	0.42 037	9.97 073	5	12
49	9.55 069	33	9.58 001	38	0.41 999	9.97 068	5	11
50	9.55 102	34	9.58 039	38	0.41 961	9.97 063	4	10
51	9.55 136	33	9.58 077	38	0.41 923	9.97 059	5	9
52	9.55 169	33	9.58 115	38	0.41 885	9.97 054	5	8
53	9.55 202	33	9.58 153	38	0.41 847	9.97 049	5	7
54	9.55 235	33	9.58 191	38	0.41 809	9.97 044	5	6
55	9.55 268	33	9.58 229	38	0.41 771	9.97 039	4	5
56	9.55 301	33	9.58 267	37	0.41 733	9.97 035	5	4
57	9.55 334	33	9.58 304	38	0.41 696	9.97 030	5	3
58	9.55 367	33	9.58 342	38	0.41 658	9.97 025	5	2
59	9.55 400	33	9.58 380	38	0.41 620	9.97 020	5	1
60	9.55 433		9.58 418		0.41 582	9.97 015		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	'

Prop. Pts.

	40	39
.1	4.0	3.9
.2	8.0	7.8
.3	12.0	11.7
.4	16.0	15.6
.5	20.0	19.5
.6	24.0	23.4
.7	28.0	27.3
.8	32.0	31.2
.9	36.0	35.1

	38	37
.1	3.8	3.7
.2	7.6	7.4
.3	11.4	11.1
.4	15.2	14.8
.5	19.0	18.5
.6	22.8	22.2
.7	26.6	25.9
.8	30.4	29.6
.9	34.2	33.3

	35
.1	3.5
.2	7.0
.3	10.5
.4	14.0
.5	17.5
.6	21.0
.7	24.5
.8	28.0
.9	31.5

	34	33
.1	3.4	3.3
.2	6.8	6.6
.3	10.2	9.9
.4	13.6	13.2
.5	17.0	16.5
.6	20.4	19.8
.7	23.8	23.1
.8	27.2	26.4
.9	30.6	29.7

	5	4
.1	0.5	0.4
.2	1.0	0.8
.3	1.5	1.2
.4	2.0	1.6
.5	2.5	2.0
.6	3.0	2.4
.7	3.5	2.8
.8	4.0	3.2
.9	4.5	3.6

Prop. Pts.

69°

21°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.55 433	33	9.58 418	37	0.41 582	9.97 015	5	**60**
1	9.55 466	33	9.58 455	38	0.41 545	9.97 010	5	59
2	9.55 499	33	9.58 493	38	0.41 507	9.97 005	4	58
3	9.55 532	32	9.58 531	38	0.41 469	9.97 001	5	57
4	9.55 564	33	9.58 569	37	0.41 431	9.96 996	5	56
5	9.55 597	33	9.58 606	38	0.41 394	9.96 991	5	55
6	9.55 630	33	9.58 644	37	0.41 356	9.96 986	5	54
7	9.55 663	32	9.58 681	38	0.41 319	9.96 981	5	53
8	9.55 695	33	9.58 719	38	0.41 281	9.96 976	5	52
9	9.55 728	33	9.58 757	37	0.41 243	9.96 971	5	51
10	9.55 761	32	9.58 794	38	0.41 206	9.96 966	4	**50**
11	9.55 793	33	9.58 832	37	0.41 168	9.96 962	5	49
12	9.55 826	32	9.58 869	38	0.41 131	9.96 957	5	48
13	9.55 858	33	9.58 907	37	0.41 093	9.96 952	5	47
14	9.55 891	32	9.58 944	37	0.41 056	9.96 947	5	46
15	9.55 923	33	9.58 981	38	0.41 019	9.96 942	5	45
16	9.55 956	32	9.59 019	37	0.40 981	9.96 937	5	44
17	9.55 988	33	9.59 056	38	0.40 944	9.96 932	5	43
18	9.56 021	32	9.59 094	37	0.40 906	9.96 927	5	42
19	9.56 053	32	9.59 131	37	0.40 869	9.96 922	5	41
20	9.56 085	33	9.59 168	37	0.40 832	9.96 917	5	**40**
21	9.56 118	32	9.59 205	38	0.40 795	9.96 912	5	39
22	9.56 150	32	9.59 243	37	0.40 757	9.96 907	4	38
23	9.56 182	33	9.59 280	37	0.40 720	9.96 903	5	37
24	9.56 215	32	9.59 317	37	0.40 683	9.96 898	5	36
25	9.56 247	32	9.59 354	37	0.40 646	9.96 893	5	35
26	9.56 279	32	9.59 391	38	0.40 609	9.96 888	5	34
27	9.56 311	32	9.59 429	37	0.40 571	9.96 883	5	33
28	9.56 343	32	9.59 466	37	0.40 534	9.96 878	5	32
29	9.56 375	33	9.59 503	37	0.40 497	9.96 873	5	31
30	9.56 408	32	9.59 540	37	0.40 460	9.96 868	5	**30**
31	9.56 440	32	9.59 577	37	0.40 423	9.96 863	5	29
32	9.56 472	32	9.59 614	37	0.40 386	9.96 858	5	28
33	9.56 504	32	9.59 651	37	0.40 349	9.96 853	5	27
34	9.56 536	32	9.59 688	37	0.40 312	9.96 848	5	26
35	9.56 568	31	9.59 725	37	0.40 275	9 96 843	5	25
36	9.56 599	32	9.59 762	37	0.40 238	9.96 838	5	24
37	9.56 631	32	9.59 799	36	0.40 201	9.96 833	5	23
38	9.56 663	32	9.59 835	37	0.40 165	9.96 828	5	22
39	9.56 695	32	9.59 872	37	0.40 128	9.96 823	5	21
40	9.56 727	32	9.59 909	37	0.40 091	9.96 818	5	**20**
41	9.56 759	31	9.59 946	37	0.40 054	9.96 813	5	19
42	9.56 790	32	9.59 983	36	0.40 017	9.96 808	5	18
43	9.56 822	32	9.60 019	37	0.39 981	9.96 803	5	17
44	9.56 854	32	9.60 056	37	0.39 944	9.96 798	5	16
45	9.56 886	31	9.60 093	37	0.39 907	9.96 793	5	15
46	9.56 917	32	9.60 130	36	0.39 870	9.96 788	5	14
47	9.56 949	31	9.60 166	37	0.39 834	9.96 783	5	13
48	9.56 980	32	9.60 203	37	0.39 797	9.96 778	6	12
49	9.57 012	32	9.60 240	36	0.39 760	9.96 772	5	11
50	9.57 044	31	9.60 276	37	0.39 724	9.96 767	5	**10**
51	9.57 075	32	9.60 313	36	0.39 687	9.96 762	5	9
52	9.57 107	31	9.60 349	37	0.39 651	9.96 757	5	8
53	9.57 138	31	9.60 386	36	0.39 614	9.96 752	5	7
54	9.57 169	32	9.60 422	37	0.39 578	9.96 747	5	6
55	9.57 201	31	9.60 459	36	0.39 541	9.96 742	5	5
56	9.57 232	32	9.60 495	37	0.39 505	9.96 737	5	4
57	9.57 264	31	9.60 532	36	0.39 468	9.96 732	5	3
58	9.57 295	31	9.60 568	37	0.39 432	9.96 727	5	2
59	9.57 326	32	9.60 605	36	0.39 395	9.96 722	5	1
60	9.57 358		9.60 641		0.39 359	9.96 717		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	38	37
.1	3.8	3.7
.2	7.6	7.4
.3	11.4	11.1
.4	15.2	14.8
.5	19.0	18.5
.6	22.8	22.2
.7	26.6	25.9
.8	30.4	29.6
.9	34.2	33.3

	36	33
.1	3.6	3.3
.2	7.2	6.6
.3	10.8	9.9
.4	14.4	13.2
.5	18.0	16.5
.6	21.6	19.8
.7	25.2	23.1
.8	28.8	26.4
.9	32.4	29.7

	32
.1	3.2
.2	6.4
.3	9.6
.4	12.8
.5	16.0
.6	19.2
.7	22.4
.8	25.6
.9	28.8

	31	6
.1	3.1	0.6
.2	6.2	1.2
.3	9.3	1.8
.4	12.4	2.4
.5	15.5	3.0
.6	18.6	3.6
.7	21.7	4.2
.8	24.8	4.8
.9	27.9	5.4

	5	4
.1	0.5	0.4
.2	1.0	0.8
.3	1.5	1.2
.4	2.0	1.6
.5	2.5	2.0
.6	3.0	2.4
.7	3.5	2.8
.8	4.0	3.2
.9	4.5	3.6

Prop. Pts.

68°

22°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.57 358	31	9.60 641	36	0.39 359	9.96 717	6	**60**
1	9.57 389	31	9.60 677	37	0.39 323	9.96 711	5	59
2	9.57 420	31	9.60 714	36	0.39 286	9.96 706	5	58
3	9.57 451	31	9.60 750	36	0.39 250	9.96 701	5	57
4	9.57 482	32	9.60 786	37	0.39 214	9.96 696	5	56
5	9.57 514	31	9.60 823	36	0.39 177	9.96 691	5	55
6	9.57 545	31	9.60 859	36	0.39 141	9.96 686	5	54
7	9.57 576	31	9.60 895	36	0.39 105	9.96 681	5	53
8	9.57 607	31	9.60 931	36	0.39 069	9.96 676	6	52
9	9.57 638	31	9.60 967	37	0.39 033	9.96 670	5	51
10	9.57 669	31	9.61 004	36	0.38 996	9.96 665	5	**50**
11	9.57 700	31	9.61 040	36	0.38 960	9.96 660	5	49
12	9.57 731	31	9.61 076	36	0.38 924	9.96 655	5	48
13	9.57 762	31	9.61 112	36	0.38 888	9.96 650	5	47
14	9.57 793	31	9.61 148	36	0.38 852	9.96 645	5	46
15	9.57 824	31	9.61 184	36	0.38 816	9.96 640	6	45
16	9.57 855	30	9.61 220	36	0.38 780	9.96 634	5	44
17	9.57 885	31	9.61 256	36	0.38 744	9.96 629	5	43
18	9.57 916	31	9.61 292	36	0.38 708	9.96 624	5	42
19	9.57 947	31	9.61 328	36	0.38 672	9.96 619	5	41
20	9.57 978	30	9.61 364	36	0.38 636	9.96 614	6	**40**
21	9.58 008	31	9.61 400	36	0.38 600	9.96 608	5	39
22	9.58 039	31	9.61 436	36	0.38 564	9.96 603	5	38
23	9.58 070	31	9.61 472	36	0.38 528	9.96 598	5	37
24	9.58 101	30	9.61 508	36	0.38 492	9.96 593	5	36
25	9.58 131	31	9.61 544	35	0.38 456	9.96 588	6	35
26	9.58 162	30	9.61 579	36	0.38 421	9.96 582	5	34
27	9.58 192	31	9.61 615	36	0.38 385	9.96 577	5	33
28	9.58 223	30	9.61 651	36	0.38 349	9.96 572	5	32
29	9.58 253	31	9.61 687	35	0.38 313	9.96 567	5	31
30	9.58 284	30	9.61 722	36	0.38 278	9.96 562	6	**30**
31	9.58 314	31	9.61 758	36	0.38 242	9.96 556	5	29
32	9.58 345	30	9.61 794	36	0.38 206	9.96 551	5	28
33	9.58 375	31	9.61 830	35	0.38 170	9.96 546	5	27
34	9.58 406	30	9.61 865	36	0.38 135	9.96 541	6	26
35	9.58 436	31	9.61 901	35	0.38 099	9.96 535	5	25
36	9.58 467	30	9.61 936	36	0.38 064	9.96 530	5	24
37	9.58 497	30	9.61 972	36	0.38 028	9.96 525	5	23
38	9.58 527	30	9.62 008	35	0.37 992	9.96 520	6	22
39	9.58 557	31	9.62 043	36	0.37 957	9.96 514	5	21
40	9.58 588	30	9.62 079	35	0.37 921	9.96 509	5	**20**
41	9.58 618	30	9.62 114	36	0.37 886	9.96 504	6	19
42	9.58 648	30	9.62 150	35	0.37 850	9.96 498	5	18
43	9.58 678	31	9.62 185	36	0.37 815	9.96 493	5	17
44	9.58 709	30	9.62 221	35	0.37 779	9.96 488	5	16
45	9.58 739	30	9.62 256	36	0.37 744	9.96 483	6	15
46	9.58 769	30	9.62 292	35	0.37 708	9.96 477	5	14
47	9.58 799	30	9.62 327	35	0.37 673	9.96 472	5	13
48	9.58 829	30	9.62 362	36	0.37 638	9.96 467	6	12
49	9.58 859	30	9.62 398	35	0.37 602	9.96 461	5	11
50	9.58 889	30	9.62 433	35	0.37 567	9.96 456	5	**10**
51	9.58 919	30	9.62 468	36	0.37 532	9.96 451	6	9
52	9.58 949	30	9.62 504	35	0.37 496	9.96 445	5	8
53	9.58 979	30	9.62 539	35	0.37 461	9.96 440	5	7
54	9.59 009	30	9.62 574	35	0.37 426	9.96 435	6	6
55	9.59 039	30	9.62 609	36	0.37 391	9.96 429	5	5
56	9.59 069	29	9.62 645	35	0.37 355	9.96 424	5	4
57	9.59 098	30	9.62 680	35	0.37 320	9.96 419	6	3
58	9.59 128	30	9.62 715	35	0.37 285	9.96 413	5	2
59	9.59 158	30	9.62 750	35	0.37 250	9.96 408	5	1
60	9.59 188		9.62 785		0.37 215	9.96 403		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	37	36
.1	3.7	3.6
.2	7.4	7.2
.3	11.1	10.8
.4	14.8	14.4
.5	18.5	18.0
.6	22.2	21.6
.7	25.9	25.2
.8	29.6	28.8
.9	33.3	32.4

	35
.1	3.5
.2	7.0
.3	10.5
.4	14.0
.5	17.5
.6	21.0
.7	24.5
.8	28.0
9	31.5

	32	31
.1	3.2	3.1
.2	6.4	6.2
.3	9.6	9.3
.4	12.8	12.4
.5	16.0	15.5
.6	19.2	18.6
.7	22.4	21.7
.8	25.6	24.8
.9	28.8	27.9

	30	29
.1	3.0	2.9
.2	6.0	5.8
.3	9.0	8.7
.4	12.0	11.6
.5	15.0	14.5
.6	18.0	17.4
.7	21.0	20.3
.8	24.0	23.2
.9	27.0	26.1

	6	5
.1	0.6	0.5
.2	1.2	1.0
.3	1.8	1.5
.4	2.4	2.0
.5	3.0	2.5
.6	3.6	3.0
.7	4.2	3.5
.8	4.8	4.0
.9	5.4	4.5

Prop. Pts.

67°

23°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.59 188	30	9.62 785	35	0.37 215̄	9.96 403	6	**60**
1	9.59 218	29	9.62 820	35	0.37 180	9.96 397	5	59
2	9.59 247	30	9.62 855	35	0.37 145̄	9.96 392	5	58
3	9.59 277	30	9.62 890	36	0.37 110	9.96 387	6	57
4	9.59 307	29	9.62 926	35	0.37 074	9.96 381	5	56
5	9.59 336	30	9.62 961	35	0.37 039	9.96 376	6	55
6	9.59 366	30	9.62 996	35	0.37 004	9.96 370	5	54
7	9.59 396	29	9.63 031	35	0.36 969	9.96 365̄	5	53
8	9.59 425	30	9.63 066	35	0.36 934	9.96 360	6	52
9	9.59 455̄	29	9.63 101	34	0.36 899	9.96 354	5	51
10	9.59 484	30	9.63 135	35	0.36 865̄	9.96 349	6	**50**
11	9.59 514	29	9.63 170	35	0.36 830	9.96 343	5	49
12	9.59 543	30	9.63 205	35	0.36 795̄	9.96 338	5	48
13	9.59 573	29	9.63 240	35	0.36 760	9.96 333	6	47
14	9.59 602	30	9.63 275	35	0.36 725̄	9.96 327	5	46
15	9.59 632	29	9.63 310	35	0.36 690	9.96 322	6	45
16	9.59 661	29	9.63 345̄	34	0.36 655	9.96 316	5	44
17	9.59 690	30	9.63 379	35	0.36 621	9.96 311	6	43
18	9.59 720	29	9.63 414	35	0.36 586	9.96 305	5	42
19	9.59 749	29	9.63 449	35	0.36 551	9.96 300	6	41
20	9.59 778	30	9.63 484	35	0.36 516	9.96 294	5	**40**
21	9.59 808	29	9.63 519	34	0.36 481	9.96 289	5	39
22	9.59 837	29	9.63 553	35	0.36 447	9.96 284	6	38
23	9.59 866	29	9.63 588	35	0.36 412	9.96 278	5	37
24	9.59 895	29	9.63 623	34	0.36 377	9.96 273	6	36
25	9.59 924	30	9.63 657	35	0.36 343	9.96 267	5	35
26	9.59 954	29	9.63 692	34	0.36 308	9.96 262	6	34
27	9.59 983	29	9.63 726	35	0.36 274	9.96 256	5	33
28	9.60 012	29	9.63 761	35	0.36 239	9.96 251	6	32
29	9.60 041	29	9.63 796	34	0.36 204	9.96 245	5	31
30	9.60 070	29	9.63 830	35	0.36 170	9.96 240	6	**30**
31	9.60 099	29	9.63 865̄	34	0.36 135	9.96 234	5	29
32	9.60 128	29	9.63 899	35	0.36 101	9.96 229	6	28
33	9.60 157	29	9.63 934	34	0.36 066	9.96 223	5	27
34	9.60 186	29	9.63 968	35	0.36 032	9.96 218	6	26
35	9.60 215̄	29	9.64 003	34	0.35 997	9.96 212	5	25
36	9.60 244	29	9.64 037	35	0.35 963	9.96 207	6	24
37	9.60 273	29	9.64 072	34	0.35 928	9.96 201	5	23
38	9.60 302	29	9.64 106	34	0.35 894	9.96 196	6	22
39	9.60 331	28	9.64 140	35	0.35 860	9.96 190	5	21
40	9.60 359	29	9.64 175̄	34	0.35 825	9.96 185̄	6	**20**
41	9.60 388	29	9.64 209	34	0.35 791	9.96 179	5	19
42	9.60 417	29	9.64 243	35	0.35 757	9.96 174	6	18
43	9.60 446	28	9.64 278	34	0.35 722	9.96 168	6	17
44	9.60 474	29	9.64 312	34	0.35 688	9.96 162	5	16
45	9.60 503	29	9.64 346	35	0.35 654	9.96 157	6	15
46	9.60 532	29	9.64 381	34	0.35 619	9.96 151	5	14
47	9.60 561	28	9.64 415̄	34	0.35 585	9.96 146	6	13
48	9.60 589	29	9.64 449	34	0.35 551	9.96 140	5	12
49	9.60 618	28	9.64 483	34	0.35 517	9.96 135̄	6	11
50	9.60 646	29	9.64 517	35	0.35 483	9.96 129	6	**10**
51	9.60 675	29	9.64 552	34	0.35 448	9.96 123	5	9
52	9.60 704	28	9.64 586	34	0.35 414	9.96 118	6	8
53	9.60 732	29	9.64 620	34	0.35 380	9.96 112	5	7
54	9.60 761	28	9.64 654	34	0.35 346	9.96 107	6	6
55	9.60 789	29	9.64 688	34	0.35 312	9.96 101	6	5
56	9.60 818	28	9.64 722	34	0.35 278	9.96 095	5	4
57	9.60 846	29	9.64 756	34	0.35 244	9.96 090	6	3
58	9.60 875̄	28	9.64 790	34	0.35 210	9.96 084	5	2
59	9.60 903	28	9.64 824	34	0.35 176	9.96 079	6	1
60	9.60 931		9.64 858		0.35 142	9.96 073		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	36	35
.1	3.6	3.5
.2	7.2	7.0
.3	10.8	10.5
.4	14.4	14.0
.5	18.0	17.5
.6	21.6	21.0
.7	25.2	24.5
.8	28.8	28.0
.9	32.4	31.5

	34
.1	3.4
.2	6.8
.3	10.2
.4	13.6
.5	17.0
.6	20.4
.7	23.8
.8	27.2
.9	30.6

	30	29
.1	3.0	2.9
.2	6.0	5.8
.3	9.0	8.7
.4	12.0	11.6
.5	15.0	14.5
.6	18.0	17.4
.7	21.0	20.3
.8	24.0	23.2
.9	27.0	26.1

	28
.1	2.8
.2	5.6
.3	8.4
.4	11.2
.5	14.0
.6	16.8
.7	19.6
.8	22.4
.9	25.2

	6	5
.1	0.6	0.5
.2	1.2	1.0
.3	1.8	1.5
.4	2.4	2.0
.5	3.0	2.5
.6	3.6	3.0
.7	4.2	3.5
.8	4.8	4.0
.9	5.4	4.5

Prop. Pts.

66°

24°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.60 931	29	9.64 858	34	0.35 142	9.96 073	6	**60**
1	9.60 960	28	9.64 892	34	0.35 108	9.96 067	5	59
2	9.60 988	28	9.64 926	34	0.35 074	9.96 062	6	58
3	9.61 016	29	9.64 960	34	0.35 040	9.96 056	6	57
4	9.61 04$\bar{5}$	28	9.64 994	34	0.35 006	9.96 050	5	56
5	9.61 073	28	9.65 028	34	0.34 972	9.96 04$\bar{5}$	6	55
6	9.61 101	28	9.65 062	34	0.34 938	9.96 039	5	54
7	9.61 129	29	9.65 096	34	0.34 904	9.96 034	6	53
8	9.61 158	28	9.65 130	34	0.34 870	9.96 028	6	52
9	9.61 186	28	9.65 164	33	0.34 836	9.96 022	5	51
10	9.61 214	28	9.65 197	34	0.34 803	9.96 017	6	**50**
11	9.61 242	28	9.65 231	34	0.34 769	9.96 011	6	49
12	9.61 270	28	9.65 265	34	0.34 73$\bar{5}$	9.96 005	5	48
13	9.61 298	28	9.65 299	34	0.34 701	9.96 000	6	47
14	9.61 326	28	9.65 333	33	0.34 667	9.95 994	6	46
15	9.61 354	28	9.65 366	34	0.34 634	9.95 988	6	45
16	9.61 382	29	9.65 400	34	0.34 600	9.95 982	5	44
17	9.61 411	27	9.65 434	33	0.34 566	9.95 977	6	43
18	9.61 438	28	9.65 467	34	0.34 533	9.95 971	6	42
19	9.61 466	28	9.65 501	34	0.34 499	9.95 965	5	41
20	9.61 494	28	9.65 53$\bar{5}$	33	0.34 465	9.95 960	6	**40**
21	9.61 522	28	9.65 568	34	0.34 432	9.95 954	6	39
22	9.61 550	28	9.65 602	34	0.34 398	9.95 948	6	38
23	9.61 578	28	9.65 636	33	0.34 364	9.95 942	5	37
24	9.61 606	28	9.65 669	34	0.34 331	9.95 937	6	36
25	9.61 634	28	9.65 703	33	0.34 297	9.95 931	6	35
26	9.61 662	27	9.65 736	34	0.34 264	9.95 925	5	34
27	9.61 689	28	9.65 770	33	0.34 230	9.95 920	6	33
28	9.61 717	28	9.65 803	34	0.34 197	9.95 914	6	32
29	9.61 74$\bar{5}$	28	9.65 837	33	0.34 163	9.95 908	6	31
30	9.61 773	27	9.65 870	34	0.34 130	9.95 902	5	**30**
31	9.61 800	28	9.65 904	33	0.34 096	9.95 897	6	29
32	9.61 828	28	9.65 937	34	0.34 063	9.95 891	6	28
33	9.61 856	27	9.65 971	33	0.34 029	9.95 88$\bar{5}$	6	27
34	9.61 883	28	9.66 004	34	0.33 996	9.95 879	6	26
35	9.61 911	28	9.66 038	33	0.33 962	9.95 873	5	25
36	9.61 939	27	9.66 071	33	0.33 929	9.95 868	6	24
37	9.61 966	28	9.66 104	34	0.33 896	9.95 862	6	23
38	9.61 994	27	9.66 138	33	0.33 862	9.95 856	6	22
39	9.62 021	28	9.66 171	33	0.33 829	9.95 850	6	21
40	9.62 049	27	9.66 204	34	0.33 796	9.95 844	5	**20**
41	9.62 076	28	9.66 238	33	0.33 762	9.95 839	6	19
42	9.62 104	27	9.66 271	33	0.33 729	9.95 833	6	18
43	9.62 131	28	9.66 304	33	0.33 696	9.95 827	6	17
44	9.62 159	27	9.66 337	34	0.33 663	9.95 821	6	16
45	9.62 186	28	9.66 371	33	0.33 629	9.95 815	5	15
46	9.62 214	27	9.66 404	33	0.33 596	9.95 810	6	14
47	9.62 241	27	9.66 437	33	0.33 563	9.95 804	6	13
48	9.62 268	28	9.66 470	33	0.33 530	9.95 798	6	12
49	9.62 296	27	9.66 503	34	0.33 497	9.95 792	6	11
50	9.62 323	27	9.66 537	33	0.33 463	9.95 786	6	**10**
51	9.62 350	27	9.66 570	33	0.33 430	9.95 780	5	9
52	9.62 377	28	9.66 603	33	0.33 397	9.95 77$\bar{5}$	6	8
53	9.62 40$\bar{5}$	27	9.66 636	33	0.33 364	9.95 769	6	7
54	9.62 432	27	9.66 669	33	0.33 331	9.95 763	6	6
55	9.62 459	27	9.66 702	33	0.33 298	9.95 757	6	5
56	9.62 486	27	9.66 735	33	0.33 26$\bar{5}$	9.95 751	6	4
57	9.62 513	28	9.66 768	33	0.33 232	9.95 745	6	3
58	9.62 541	27	9.66 801	33	0.33 199	9.95 739	6	2
59	9.62 568	27	9.66 834	33	0.33 166	9.95 733	5	1
60	9.62 59$\bar{5}$		9.66 867		0.33 133	9.95 728		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	34	33
.1	3.4	3.3
.2	6.8	6.6
.3	10.2	9.9
.4	13.6	13.2
.5	17.0	16.5
.6	20.4	19.8
.7	23.8	23.1
.8	27.2	26.4
.9	30.6	29.7

	29
.1	2.9
.2	5.8
.3	8.7
.4	11.6
.5	14.5
.6	17.4
.7	20.3
.8	23.2
.9	26.1

	28
.1	2.8
.2	5.6
.3	8.4
.4	11.2
.5	14.0
.6	16.8
.7	19.6
.8	22.4
.9	25.2

	27
.1	2.7
.2	5.4
.3	8.1
.4	10.8
.5	13.5
.6	16.2
.7	18.9
.8	21.6
.9	24.3

	6	5
.1	0.6	0.5
.2	1.2	1.0
.3	1.8	1.5
.4	2.4	2.0
.5	3.0	2.5
.6	3.6	3.0
.7	4.2	3.5
.8	4.8	4.0
.9	5.4	4.5

Prop. Pts.

65°

25°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.62 595̄	27	9.66 867	33	0.33 133	9.95 728	6	60
1	9.62 622	27	9.66 900	33	0.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	0.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	0.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	0.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	0.32 968	9.95 698	6	55
6	9.62 757	27	9.67 065̄	33	0.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	0.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	32	0.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	33	0.32 837	9.95 674	6	51
10	9.62 865̄	27	9.67 196	33	0.32 804	9.95 668	5	50
11	9.62 892	26	9.67 229	33	0.32 771	9.95 663	6	49
12	9.62 918	27	9.67 262	33	0.32 738	9.95 657	6	48
13	9.62 945	27	9.67 295̄	32	0.32 705	9.95 651	6	47
14	9.62 972	27	9.67 327	33	0.32 673	9.95 645̄	6	46
15	9.62 999	27	9.67 360	33	0.32 640	9.95 639	6	45
16	9.63 026	26	9.67 393	33	0.32 607	9.95 633	6	44
17	9.63 052	27	9.67 426	32	0.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	33	0.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	0.32 509	9.95 615̄	6	41
20	9.63 133	26	9.67 524	32	0.32 476	9.95 609	6	40
21	9.63 159	27	9.67 556	33	0.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	0.32 411	9.95 597	6	38
23	9.63 213	26	9.67 622	32	0.32 378	9.95 591	6	37
24	9.63 239	27	9.67 654	33	0.32 346	9.95 585̄	6	36
25	9.63 266	26	9.67 687	32	0.32 313	9.95 579	6	35
26	9.63 292	27	9.67 719	33	0.32 281	9.95 573	6	34
27	9.63 319	26	9.67 752	33	0.32 248	9.95 567	6	33
28	9.63 345	27	9.67 785̄	32	0.32 215	9.95 561	6	32
29	9.63 372	26	9.67 817	33	0.32 183	9.95 555̄	6	31
30	9.63 398	27	9.67 850	32	0.32 150	9.95 549	6	30
31	9.63 425̄	26	9.67 882	33	0.32 118	9.95 543	6	29
32	9.63 451	27	9.67 915̄	32	0.32 085	9.95 537	6	28
33	9.63 478	26	9.67 947	33	0.32 053	9.95 531	6	27
34	9.63 504	27	9.67 980	32	0.32 020	9.95 525̄	6	26
35	9.63 531	26	9.68 012	32	0.31 988	9.95 519	6	25
36	9.63 557	26	9.68 044	33	0.31 956	9.95 513	6	24
37	9.63 583	27	9.68 077	32	0.31 923	9.95 507	7	23
38	9.63 610	26	9.68 109	33	0.31 891	9.95 500	6	22
39	9.63 636	26	9.68 142	32	0.31 858	9.95 494	6	21
40	9.63 662	27	9.68 174	32	0.31 826	9.95 488	6	20
41	9.63 689	26	9.68 206	33	0.31 794	9.95 482	6	19
42	9.63 715̄	26	9.68 239	32	0.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	0.31 729	9.95 470	6	17
44	9.63 767	27	9.68 303	33	0.31 697	9.95 464	6	16
45	9.63 794	26	9.68 336	32	0.31 664	9.95 458	6	15
46	9.63 820	26	9.68 368	32	0.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	0.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	33	0.31 568	9.95 440	6	12
49	9.63 898	26	9.68 465̄	32	0.31 535	9.95 434	7	11
50	9.63 924	26	9.68 497	32	0.31 503	9.95 427	6	10
51	9.63 950	26	9.68 529	32	0.31 471	9.95 421	6	9
52	9.63 976	26	9.68 561	32	0.31 439	9.95 415	6	8
53	9.64 002	26	9.68 593	33	0.31 407	9.95 409	6	7
54	9.64 028	26	9.68 626	32	0.31 374	9.95 403	6	6
55	9.64 054	26	9.68 658	32	0.31 342	9.95 397	6	5
56	9.64 080	26	9.68 690	32	0.31 310	9.95 391	7	4
57	9.64 106	26	9.68 722	32	0.31 278	9.95 384	6	3
58	9.64 132	26	9.68 754	32	0.31 246	9.95 378	6	2
59	9.64 158	26	9.68 786	32	0.31 214	9.95 372	6	1
60	9.64 184		9.68 818		0.31 182	9.95 366		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	33	32	27	26	7	6	5
.1	3.3	3.2	2.7	2.6	0.7	0.6	0.5
.2	6.6	6.4	5.4	5.2	1.4	1.2	1.0
.3	9.9	9.6	8.1	7.8	2.1	1.8	1.5
.4	13.2	12.8	10.8	10.4	2.8	2.4	2.0
.5	16.5	16.0	13.5	13.0	3.5	3.0	2.5
.6	19.8	19.2	16.2	15.6	4.2	3.6	3.0
.7	23.1	22.4	18.9	18.2	4.9	4.2	3.5
.8	26.4	25.6	21.6	20.8	5.6	4.8	4.0
.9	29.7	28.8	24.3	23.4	6.3	5.4	4.5

Prop. Pts.

64°

26°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	**60**
1	9.64 210	26	9.68 850	32	0.31 150	9.95 360	6	59
2	9.64 236	26	9.68 882	32	0.31 118	9.95 354	6	58
3	9.64 262	26	9.68 914	32	0.31 086	9.95 348	7	57
4	9.64 288	25	9.68 946	32	0.31 054	9.95 341	6	56
5	9.64 313	26	9.68 978	32	0.31 022	9.95 335	6	55
6	9.64 339	26	9.69 010	32	0.30 990	9.95 329	6	54
7	9.64 365	26	9.69 042	32	0.30 958	9.95 323	6	53
8	9.64 391	26	9.69 074	32	0.30 926	9.95 317	7	52
9	9.64 417	25	9.69 106	32	0.30 894	9.95 310	6	51
10	9.64 442	26	9.69 138	32	0.30 862	9.95 304	6	**50**
11	9.64 468	26	9.69 170	32	0.30 830	9.95 298	6	49
12	9.64 494	25	9.69 202	32	0.30 798	9.95 292	6	48
13	9.64 519	26	9.69 234	32	0.30 766	9.95 286	7	47
14	9.64 545	26	9.69 266	32	0.30 734	9.95 279	6	46
15	9.64 571	25	9.69 298	31	0.30 702	9.95 273	6	45
16	9.64 596	26	9.69 329	32	0.30 671	9.95 267	6	44
17	9.64 622	25	9.69 361	32	0.30 639	9.95 261	7	43
18	9.64 647	26	9.69 393	32	0.30 607	9.95 254	6	42
19	9.64 673	25	9.69 425	32	0.30 575	9.95 248	6	41
20	9.64 698	26	9.69 457	31	0.30 543	9.95 242	6	**40**
21	9.64 724	25	9.69 488	32	0.30 512	9.95 236	7	39
22	9.64 749	26	9.69 520	32	0.30 480	9.95 229	6	38
23	9.64 775	25	9.69 552	32	0.30 448	9.95 223	6	37
24	9.64 800	26	9.69 584	31	0.30 416	9.95 217	6	36
25	9.64 826	25	9.69 615	32	0.30 385	9.95 211	7	35
26	9.64 851	26	9.69 647	32	0.30 353	9.95 204	6	34
27	9.64 877	25	9.69 679	31	0.30 321	9.95 198	6	33
28	9.64 902	25	9.69 710	32	0.30 290	9.95 192	7	32
29	9.64 927	26	9.69 742	32	0.30 258	9.95 185	6	31
30	9.64 953	25	9.69 774	31	0.30 226	9.95 179	6	**30**
31	9.64 978	25	9.69 805	32	0.30 195	9.95 173	6	29
32	9.65 003	26	9.69 837	31	0.30 163	9.95 167	7	28
33	9.65 029	25	9.69 868	32	0.30 132	9.95 160	6	27
34	9.65 054	25	9.69 900	32	0.30 100	9.95 154	6	26
35	9.65 079	25	9.69 932	31	0.30 068	9.95 148	7	25
36	9.65 104	26	9.69 963	32	0.30 037	9.95 141	6	24
37	9.65 130	25	9.69 995	31	0.30 005	9.95 135	6	23
38	9.65 155	25	9.70 026	32	0.29 974	9.95 129	7	22
39	9.65 180	25	9.70 058	31	0.29 942	9.95 122	6	21
40	9.65 205	25	9.70 089	32	0.29 911	9.95 116	6	**20**
41	9.65 230	25	9.70 121	31	0.29 879	9.95 110	7	19
42	9.65 255	26	9.70 152	32	0.29 848	9.95 103	6	18
43	9.65 281	25	9.70 184	31	0.29 816	9.95 097	7	17
44	9.65 306	25	9.70 215	32	0.29 785	9.95 090	6	16
45	9.65 331	25	9.70 247	31	0.29 753	9.95 084	6	15
46	9.65 356	25	9.70 278	31	0.29 722	9.95 078	7	14
47	9.65 381	25	9.70 309	32	0.29 691	9.95 071	6	13
48	9.65 406	25	9.70 341	31	0.29 659	9.95 065	6	12
49	9.65 431	25	9.70 372	32	0.29 628	9.95 059	7	11
50	9.65 456	25	9.70 404	31	0.29 596	9.95 052	6	**10**
51	9.65 481	25	9.70 435	31	0.29 565	9.95 046	7	9
52	9.65 506	25	9.70 466	32	0.29 534	9.95 039	6	8
53	9.65 531	25	9.70 498	31	0.29 502	9.95 033	6	7
54	9.65 556	24	9.70 529	31	0.29 471	9.95 027	7	6
55	9.65 580	25	9.70 560	32	0.29 440	9.95 020	6	5
56	9.65 605	25	9.70 592	31	0.29 408	9.95 014	7	4
57	9.65 630	25	9.76 623	31	0.29 377	9.95 007	6	3
58	9.65 655	25	9.70 654	31	0.29 346	9.95 001	6	2
59	9.65 680	25	9.70 685	32	0.29 315	9.94 995	7	1
60	9.65 705		9.70 717		0.29 283	9.94 988		**0**
	L. Cos.	**d.**	**L. Cotg.**	**c. d.**	**L. Tang.**	**L. Sin.**	**d.**	**′**

63°

Prop. Pts.

	32	31
.1	3.2	3.1
.2	6.4	6.2
.3	9.6	9.3
.4	12.8	12.4
.5	16.0	15.5
.6	19.2	18.6
.7	22.4	21.7
.8	25.6	24.8
.9	28.8	27.9

	26
.1	2.6
.2	5.2
.3	7.8
.4	10.4
.5	13.0
.6	15.6
.7	18.2
.8	20.8
.9	23.4

	25
.1	2.5
.2	5.0
.3	7.5
.4	10.0
.5	12.5
.6	15.0
.7	17.5
.8	20.0
.9	22.5

	24
.1	2.4
.2	4.8
.3	7.2
.4	9.6
.5	12.0
.6	14.4
.7	16.8
.8	19.2
.9	21.6

	7	6
.1	0.7	0.6
.2	1.4	1.2
.3	2.1	1.8
.4	2.8	2.4
.5	3.5	3.0
.6	4.2	3.6
.7	4.9	4.2
.8	5.6	4.8
.9	6.3	5.4

Prop. Pts.

27°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.65 705̄	24	9.70 717	31	0.29 283	9.94 988	6	**60**
1	9.65 729	25	9.70 748	31	0.29 252	9.94 982	7	59
2	9.65 754	25	9.70 779	31	0.29 221	9.94 975	6	58
3	9.65 779	25	9.70 810	31	0.29 190	9.94 969	7	57
4	9.65 804	24	9.70 841	32	0.29 159	9.94 962	6	56
5	9.65 828	25	9.70 873	31	0.29 127	9.94 956	7	55
6	9.65 853	25	9.70 904	31	0.29 096	9.94 949	6	54
7	9.65 878	24	9.70 935̄	31	0.29 065	9.94 943	7	53
8	9.65 902	25	9.70 966	31	0.29 034	9.94 936	6	52
9	9.65 927	25	9.70 997	31	0.29 003	9.94 930	7	51
10	9.65 952	24	9.71 028	31	0.28 972	9.94 923	6	**50**
11	9.65 976	25	9.71 059	31	0.28 941	9.94 917	6	49
12	9.66 001	24	9.71 090	31	0.28 910	9.94 911	7	48
13	9.66 025	25	9.71 121	32	0.28 879	9.94 904	6	47
14	9.66 050	25	9.71 153	31	0.28 847	9.94 898	7	46
15	9.66 075̄	24	9.71 184	31	0.28 816	9.94 891	6	45
16	9.66 099	25	9.71 215̄	31	0.28 785	9.94 885̄	7	44
17	9.66 124	24	9.71 246	31	0.28 754	9.94 878	7	43
18	9.66 148	25	9.71 277	31	0.28 723	9.94 871	6	42
19	9.66 173	24	9.71 308	31	0.28 692	9.94 865̄	7	41
20	9.66 197	24	9.71 339	31	0.28 661	9.94 858	6	**40**
21	9.66 221	25	9.71 370	31	0.28 630	9.94 852	7	39
22	9.66 246	24	9.71 401	30	0.28 599	9.94 845	6	38
23	9.66 270	25	9.71 431	31	0.28 569	9.94 839	7	37
24	9.66 295̄	24	9.71 462	31	0.28 538	9.94 832	6	36
25	9.66 319	24	9.71 493	31	0.28 507	9.94 826	7	35
26	9.66 343	25	9.71 524	31	0.28 476	9.94 819	6	34
27	9.66 368	24	9.71 555	31	0.28 445̄	9.94 813	7	33
28	9.66 392	24	9.71 586	31	0.28 414	9.94 806	7	32
29	9.66 416	25	9.71 617	31	0.28 383	9.94 799	6	31
30	9.66 441	24	9.71 648	31	0.28 352	9.94 793	7	**30**
31	9.66 465̄	24	9.71 679	30	0.28 321	9.94 786	6	29
32	9.66 489	24	9.71 709	31	0.28 291	9.94 780	7	28
33	9.66 513	24	9.71 740	31	0.28 260	9.94 773	6	27
34	9.66 537	25	9.71 771	31	0.28 229	9.94 767	7	26
35	9.66 562	24	9.71 802	31	0.28 198	9.94 760	7	25
36	9.66 586	24	9.71 833	30	0.28 167	9.94 753	6	24
37	9.66 610	24	9.71 863	31	0.28 137	9.94 747	7	23
38	9.66 634	24	9.71 894	31	0.28 106	9.94 740	6	22
39	9.66 658	24	9.71 925̄	30	0.28 075	9.94 734	7	21
40	9.66 682	24	9.71 955	31	0.28 045̄	9.94 727	7	**20**
41	9.66 706	25	9.71 986	31	0.28 014	9.94 720	6	19
42	9.66 731	24	9.72 017	31	0.27 983	9.94 714	7	18
43	9.66 755̄	24	9.72 048	30	0.27 952	9.94 707	7	17
44	9.66 779	24	9.72 078	31	0.27 922	9.94 700	6	16
45	9.66 803	24	9.72 109	31	0.27 891	9.94 694	7	15
46	9.66 827	24	9.72 140	30	0.27 860	9.94 687	7	14
47	9.66 851	24	9.72 170	31	0.27 830	9.94 680	6	13
48	9.66 875̄	24	9.72 201	30	0.27 799	9.94 674	7	12
49	9.66 899	23	9.72 231	31	0.27 769	9.94 667	7	11
50	9.66 922	24	9.72 262	31	0.27 738	9.94 660	6	**10**
51	9.66 946	24	9.72 293	30	0.27 707	9.94 654	7	9
52	9.66 970	24	9.72 323	31	0.27 677	9.94 647	7	8
53	9.66 994	24	9.72 354	30	0.27 646	9.94 640	6	7
54	9.67 018	24	9.72 384	31	0.27 616	9.94 634	7	6
55	9.67 042	24	9.72 415̄	30	0.27 585̄	9.94 627	7	5
56	9.67 066	24	9.72 445	31	0.27 555̄	9.94 620	6	4
57	9.67 090	23	9.72 476	30	0.27 524	9.94 614	7	3
58	9.67 113	24	9.72 506	31	0.27 494	9.94 607	7	2
59	9.67 137	24	9.72 537	30	0.27 463	9.94 600	7	1
60	9.67 161		9.72 567		0.27 433	9.94 593		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

62°

Prop. Pts.

	32	31
.1	3.2	3.1
.2	6.4	6.2
.3	9.6	9.3
.4	12.8	12.4
.5	16.0	15.5
.6	19.2	18.6
.7	22.4	21.7
.8	25.6	24.8
.9	28.8	27.9

	30
.1	3.0
.2	6.0
.3	9.0
.4	12.0
.5	15.0
.6	18.0
.7	21.0
.8	24.0
.9	27.0

	25	24
.1	2.5	2.4
.2	5.0	4.8
.3	7.5	7.2
.4	10.0	9.6
.5	12.5	12.0
.6	15.0	14.4
.7	17.5	16.8
.8	20.0	19.2
.9	22.5	21.6

	23
.1	2.3
.2	4.6
.3	6.9
.4	9.2
.5	11.5
.6	13.8
.7	16.1
.8	18.4
.9	20.7

	7	6
.1	0.7	0.6
.2	1.4	1.2
.3	2.1	1.8
.4	2.8	2.4
.5	3.5	3.0
.6	4.2	3.6
.7	4.9	4.2
.8	5.6	4.8
.9	6.3	5.4

Prop. Pts.

28°

L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
9.67 161	24	9.72 567	31	0.27 433	9.94 593	6	60
9.67 185	23	9.72 598	30	0.27 402	9.94 587	7	59
9.67 208	24	9.72 628	31	0.27 372	9.94 580	7	58
9.67 232	24	9.72 659	30	0.27 341	9.94 573	6	57
9.67 256	24	9.72 689	31	0.27 311	9.94 567	7	56
9.67 280	23	9.72 720	30	0.27 280	9.94 560	7	55
9.67 303	24	9.72 750	30	0.27 250	9.94 553	7	54
9.67 327	23	9.72 780	31	0.27 220	9.94 546	6	53
9.67 350	24	9.72 811	30	0.27 189	9.94 540	7	52
9.67 374	24	9.72 841	31	0.27 159	9.94 533	7	51
9.67 398	23	9.72 872	30	0.27 128	9.94 526	7	50
9.67 421	24	9.72 902	30	0.27 098	9.94 519	6	49
9.67 445	23	9.72 932	31	0.27 068	9.94 513	7	48
9.67 468	24	9.72 963	30	0.27 037	9.94 506	7	47
9.67 492	23	9.72 993	30	0.27 007	9.94 499	7	46
9.67 515	24	9.73 023	31	0.26 977	9.94 492	7	45
9.67 539	23	9.73 054	30	0.26 946	9.94 485	6	44
9.67 562	24	9.73 084	30	0.26 916	9.94 479	7	43
9.67 586	23	9.73 114	30	0.26 886	9.94 472	7	42
9.67 609	24	.73 144	31	0.26 856	9.94 465	7	41
9.67 633	23	9.73 175	30	0.26 825	9.94 458	7	40
9.67 656	24	9.73 205	30	0.26 795	9.94 451	6	39
9.67 680	23	9.73 235	30	0.26 765	9.94 445	7	38
9.67 703	23	9.73 265	30	0.26 735	9.94 438	7	37
9.67 726	24	9.73 295	31	0.26 705	9.94 431	7	36
9.67 750	23	9.73 326	30	0.26 674	9.94 424	7	35
9.67 773	23	9.73 356	30	0.26 644	9.94 417	7	34
9.67 796	24	9.73 386	30	0.26 614	9.94 410	6	33
9.67 820	23	9.73 416	30	0.26 584	9.94 404	7	32
9.67 843	23	9.73 446	30	0.26 554	9.94 397	7	31
9.67 866	24	9.73 476	31	0.26 524	9.94 390	7	80
9.67 890	23	9.73 507	30	0.26 493	9.94 383	7	29
9.67 913	23	9.73 537	30	0.26 463	9.94 376	7	28
9.67 936	23	9.73 567	30	0.26 433	9.94 369	7	27
9.67 959	23	9.73 597	30	0.26 403	9.94 362	7	26
9.67 982	24	9.73 627	30	0.26 373	9.94 355	6	25
9.68 006	23	9.73 657	30	0.26 343	9.94 349	7	24
9.68 029	23	9.73 687	30	0.26 313	9.94 342	7	23
9.68 052	23	9.73 717	30	0.26 283	9.94 335	7	22
9.68 075	23	9.73 747	30	0.26 253	9.94 328	7	21
9.68 098	23	9.73 777	30	0.26 223	9.94 321	7	20
9.68 121	23	9.73 807	30	0.26 193	9.94 314	7	19
9.68 144	23	9.73 837	30	0.26 163	9.94 307	7	18
9.68 167	23	9.73 867	30	0.26 133	9.94 300	7	17
9.68 190	23	9.73 897	30	0.26 103	9.94 293	7	16
9.68 213	24	9.73 927	30	0.26 073	9.94 286	7	15
9.68 237	23	9.73 957	30	0.26 043	9.94 279	6	14
9.68 260	23	9.73 987	30	0.26 013	9.94 273	7	13
9.68 283	22	9.74 017	30	0.25 983	9.94 266	7	12
9.68 305	23	9.74 047	30	0.25 953	9.94 259	7	11
9.68 328	23	9.74 077	30	0.25 923	9.94 252	7	10
9.68 351	23	9.74 107	30	0.25 893	9.94 245	7	9
9.68 374	23	9.74 137	29	0.25 863	9.94 238	7	8
9.68 397	23	9.74 166	30	0.25 834	9.94 231	7	7
9.68 420	23	9.74 196	30	0.25 804	9.94 224	7	6
9.68 443	23	9.74 226	30	0.25 774	9.94 217	7	5
9.68 466	23	9.74 256	30	0.25 744	9.94 210	7	4
9.68 489	23	9.74 286	30	0.25 714	9.94 203	7	3
9.68 512	22	9.74 316	29	0.25 684	9.94 196	7	2
9.68 534	23	9.74 345	30	0.25 655	9.94 189	7	1
9.68 557		9.74 375		0.25 625	9.94 182		0
L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	'

Prop. Pts.

	31	30	29	24	23	22	7	6
.1	3.1	3.0	2.9	2.4	2.3	2.2	0.7	0.6
.2	6.2	6.0	5.8	4.8	4.6	4.4	1.4	1.2
.3	9.3	9.0	8.7	7.2	6.9	6.6	2.1	1.8
.4	12.4	12.0	11.6	9.6	9.2	8.8	2.8	2.4
.5	15.5	15.0	14.5	12.0	11.5	11.0	3.5	3.0
.6	18.6	18.0	17.4	14.4	13.8	13.2	4.2	3.6
.7	21.7	21.0	20.3	16.8	16.1	15.4	4.9	4.2
.8	24.8	24.0	23.2	19.2	18.4	17.6	5.6	4.8
.9	27.9	27.0	26.1	21.6	20.7	19.8	6.3	5.4

Prop. Pts.

61°

29°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.68 557	23	9.74 375	30	0.25 625̄	9.94 182	7	**60**
1	9.68 580	23	9.74 405̄	30	0.25 595	9.94 175̄	7	59
2	9.68 603	22	9.74 435̄	30	0.25 565	9.94 168	7	58
3	9.68 625	23	9.74 465̄	29	0.25 535	9.94 161	7	57
4	9.68 648	23	9.74 494	30	0.25 506	9.94 154	7	56
5	9.68 671	23	9.74 524	30	0.25 476	9.94 147	7	55
6	9.68 694	22	9.74 554	29	0.25 446	9.94 140	7	54
7	9.68 716	23	9.74 583	30	0.25 417	9.94 133	7	53
8	9.68 739	23	9.74 613	30	0.25 387	9.94 126	7	52
9	9.68 762	22	9.74 643	30	0.25 357	9.94 119	7	51
10	9.68 784	23	9.74 673	29	0.25 327	9.94 112	7	**50**
11	9.68 807	22	9.74 702	30	0.25 298	9.94 105̄	7	49
12	9.68 829	23	9.74 732	30	0.25 268	9.94 098	8	48
13	9.68 852	23	9.74 762	29	0.25 238	9.94 090	7	47
14	9.68 875̄	22	9.74 791	30	0.25 209	9.94 083	7	46
15	9.68 897	23	9.74 821	30	0.25 179	9.94 076	7	45
16	9.68 920	22	9.74 851	29	0.25 149	9.94 069	7	44
17	9.68 942	23	9.74 880	30	0.25 120	9.94 062	7	43
18	9.68 965̄	22	9.74 910	29	0.25 090	9.94 055	7	42
19	9.68 987	23	9.74 939	30	0.25 061	9.94 048	7	41
20	9.69 010	22	9.74 969	29	0.25 031	9.94 041	7	**40**
21	9.69 032	23	9.74 998	30	0.25 002	9.94 034	7	39
22	9.69 055̄	22	9.75 028	30	0.24 972	9.94 027	7	38
23	9.69 077	23	9.75 058	29	0.24 942	9.94 020	8	37
24	9.69 100	22	9.75 087	30	0.24 913	9.94 012	7	36
25	9.69 122	22	9.75 117	29	0.24 883	9.94 005	7	35
26	9.69 144	23	9.75 146	30	0.24 854	9.93 998	7	34
27	9.69 167	22	9.75 176	29	0.24 824	9.93 991	7	33
28	9.69 189	23	9.75 205	30	0.24 795̄	9.93 984	7	32
29	9.69 212	22	9.75 235̄	29	0.24 765	9.93 977	7	31
30	9.69 234	22	9.75 264	30	0.24 736	9.93 970	7	**30**
31	9.69 256	23	9.75 294	29	0.24 706	9.93 963	8	29
32	9.69 279	22	9.75 323	30	0.24 677	9.93 955	7	28
33	9.69 301	22	9.75 353	29	0.24 647	9.93 948	7	27
34	9.69 323	22	9.75 382	29	0.24 618	9.93 941	7	26
35	9.69 345	23	9.75 411	30	0.24 589	9.93 934	7	25
36	9.69 368	22	9.75 441	29	0.24 559	9.93 927	7	24
37	9.69 390	22	9.75 470	30	0.24 530	9.93 920	8	23
38	9.69 412	22	9.75 500	29	0.24 500	9.93 912	7	22
39	9.69 434	22	9.75 529	29	0.24 471	9.93 905	7	21
40	9.69 456	23	9.75 558	30	0.24 442	9.93 898	7	**20**
41	9.69 479	22	9.75 588	29	0.24 412	9.93 891	7	19
42	9.69 501	22	9.75 617	30	0.24 383	9.93 884	8	18
43	9.69 523	22	9.75 647	29	0.24 353	9.93 876	7	17
44	9.69 545	22	9.75 676	29	0.24 324	9.93 869	7	16
45	9.69 567	22	9.75 705	30	0.24 295̄	9.93 862	7	15
46	9.69 589	22	9.75 735̄	29	0.24 265	9.93 855̄	8	14
47	9.69 611	22	9.75 764	29	0.24 236	9.93 847	7	13
48	9.69 633	22	9.75 793	29	0.24 207	9.93 840	7	12
49	9.69 655	22	9.75 822	30	0.24 178	9.93 833	7	11
50	9.69 677	22	9.75 852	29	0.24 148	9.93 826	7	**10**
51	9.69 699	22	9.75 881	29	0.24 119	9.93 819	8	9
52	9.69 721	22	9.75 910	29	0.24 090	9.93 811	7	8
53	9.69 743	22	9.75 939	30	0.24 061	9.93 804	7	7
54	9.69 765	22	9.75 969	29	0.24 031	9.93 797	8	6
55	9.69 787	22	9.75 998	29	0.24 002	9.93 789	7	5
56	9.69 809	22	9.76 027	29	0.23 973	9.93 782	7	4
57	9.69 831	22	9.76 056	30	0.23 944	9.93 775̄	7	3
58	9.69 853	22	9.76 086	29	0.23 914	9.93 768	8	2
59	9.69 875	22	9.76 115̄	29	0.23 885	9.93 760	7	1
60	9.69 897		9.76 144		0.23 856	9.93 753		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	30
.1	3.0
.2	6.0
.3	9.0
.4	12.0
.5	15.0
.6	18.0
.7	21.0
.8	24.0
.9	27.0

	29
.1	2.9
.2	5.8
.3	8.7
.4	11.6
.5	14.5
.6	17.4
.7	20.3
.8	23.2
.9	26.1

	23
.1	2.3
.2	4.6
.3	6.9
.4	9.2
.5	11.5
.6	13.8
.7	16.1
.8	18.4
.9	20.7

	22
.1	2.2
.2	4.4
.3	6.6
.4	8.8
.5	11.0
.6	13.2
.7	15.4
.8	17.6
.9	19.8

	8	7
.1	0.8	0.7
.2	1.6	1.4
.3	2.4	2.1
.4	3.2	2.8
.5	4.0	3.5
.6	4.8	4.2
.7	5.6	4.9
.8	6.4	5.6
.9	7.2	6.3

Prop. Pts.

60°

30°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	60
1	9.69 919	22	9.76 173	29	0.23 827	9.93 746	8	59
2	9.69 941	22	9.76 202	29	0.23 798	9.93 738	7	58
3	9.69 963	21	9.76 231	30	0.23 769	9.93 731	7	57
4	9.69 984	22	9.76 261	29	0.23 739	9.93 724	7	56
5	9.70 006	22	9.76 290	29	0.23 710	9.93 717	8	55
6	9.70 028	22	9.76 319	29	0.23 681	9.93 709	7	54
7	9.70 050	22	9.76 348	29	0.23 652	9.93 702	7	53
8	9.70 072	21	9.76 377	29	0.23 623	9.93 695	8	52
9	9.70 093	22	9.76 406	29	0.23 594	9.93 687	7	51
10	9.70 115	22	9.76 435	29	0.23 565	9.93 680	7	50
11	9.70 137	22	9.76 464	29	0.23 536	9.93 673	8	49
12	9.70 159	21	9.76 493	29	0.23 507	9.93 665	7	48
13	9.70 180	22	9.76 522	29	0.23 478	9.93 658	8	47
14	9.70 202	22	9.76 551	29	0.23 449	9.93 650	7	46
15	9.70 224	21	9.76 580	29	0.23 420	9.93 643	7	45
16	9.70 245	22	9.76 609	30	0.23 391	9.93 636	8	44
17	9.70 267	21	9.76 639	29	0.23 361	9.93 628	7	43
18	9.70 288	22	9.76 668	29	0.23 332	9.93 621	7	42
19	9.70 310	22	9.76 697	28	0.23 303	9.93 614	8	41
20	9.70 332	21	9.76 725	29	0.23 275	9.93 606	7	40
21	9.70 353	22	9.76 754	29	0.23 246	9.93 599	8	39
22	9.70 375	21	9.76 783	29	0.23 217	9.93 591	7	38
23	9.70 396	22	9.76 812	29	0.23 188	9.93 584	7	37
24	9.70 418	21	9.76 841	29	0.23 159	9.93 577	8	36
25	9.70 439	22	9.76 870	29	0.23 130	9.93 569	7	35
26	9.70 461	21	9.76 899	29	0.23 101	9.93 562	8	34
27	9.70 482	22	9.76 928	29	0.23 072	9.93 554	7	33
28	9.70 504	21	9.76 957	29	0.23 043	9.93 547	8	32
29	9.70 525	22	9.76 986	29	0.23 014	9.93 539	7	31
30	9.70 547	21	9.77 015	29	0.22 985	9.93 532	7	30
31	9.70 568	22	9.77 044	29	0.22 956	9.93 525	8	29
32	9.70 590	21	9.77 073	28	0.22 927	9.93 517	7	28
33	9.70 611	22	9.77 101	29	0.22 899	9.93 510	8	27
34	9.70 633	21	9.77 130	29	0.22 870	9.93 502	7	26
35	9.70 654	21	9.77 159	29	0.22 841	9.93 495	8	25
36	9.70 675	22	9.77 188	29	0.22 812	9.93 487	7	24
37	9.70 697	21	9.77 217	29	0.22 783	9.93 480	8	23
38	9.70 718	21	9.77 246	28	0.22 754	9.93 472	7	22
39	9.70 739	22	9.77 274	29	0.22 726	9.93 465	8	21
40	9.70 761	21	9.77 303	29	0.22 697	9.93 457	7	20
41	9.70 782	21	9.77 332	29	0.22 668	9.93 450	8	19
42	9.70 803	21	9.77 361	29	0.22 639	9.93 442	7	18
43	9.70 824	22	9.77 390	28	0.22 610	9.93 435	8	17
44	9.70 846	21	9.77 418	29	0.22 582	9.93 427	7	16
45	9.70 867	21	9.77 447	29	0.22 553	9.93 420	8	15
46	9.70 888	21	9.77 476	29	0.22 524	9.93 412	7	14
47	9.70 909	22	9.77 505	28	0.22 495	9.93 405	8	13
48	9.70 931	21	9.77 533	29	0.22 467	9.93 397	7	12
49	9.70 952	21	9.77 562	29	0.22 438	9.93 390	8	11
50	9.70 973	21	9.77 591	28	0.22 409	9.93 382	7	10
51	9.70 994	21	9.77 619	29	0.22 381	9.93 375	8	9
52	9.71 015	21	9.77 648	29	0.22 352	9.93 367	7	8
53	9.71 036	22	9.77 677	29	0.22 323	9.93 360	8	7
54	9.71 058	21	9.77 706	28	0.22 294	9.93 352	8	6
55	9.71 079	21	9.77 734	29	0.22 266	9.93 344	7	5
56	9.71 100	21	9.77 763	28	0.22 237	9.93 337	8	4
57	9.71 121	21	9.77 791	29	0.22 209	9.93 329	7	3
58	9.71 142	21	9.77 820	29	0.22 180	9.93 322	8	2
59	9.71 163	21	9.77 849	28	0.22 151	9.93 314	7	1
60	9.71 184		9.77 877		0.22 123	9.93 307		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	30	29
.1	3.0	2.9
.2	6.0	5.8
.3	9.0	8.7
.4	12.0	11.6
.5	15.0	14.5
.6	18.0	17.4
.7	21.0	20.3
.8	24.0	23.2
.9	27.0	26.1

	28
.1	2.8
.2	5.6
.3	8.4
.4	11.2
.5	14.0
.6	16.8
.7	19.6
.8	22.4
.9	25.2

	22
.1	2.2
.2	4.4
.3	6.6
.4	8.8
.5	11.0
.6	13.2
.7	15.4
.8	17.6
.9	19.8

	21
.1	2.1
.2	4.2
.3	6.3
.4	8.4
.5	10.5
.6	12.6
.7	14.7
.8	16.8
.9	18.9

	8	7
.1	0.8	0.7
.2	1.6	1.4
.3	2.4	2.1
.4	3.2	2.8
.5	4.0	3.5
.6	4.8	4.2
.7	5.6	4.9
.8	6.4	5.6
.9	7.2	6.3

Prop. Pts.

59°

31°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.71 184	21	9.77 877	29	0.22 123	9.93 307	8	60
1	9.71 205̄	21	9.77 906	29	0.22 094	9.93 299	8	59
2	9.71 226	21	9.77 935̄	28	0.22 065	9.93 291	7	58
3	9.71 247	21	9.77 963	29	0.22 037	9.93 284	8	57
4	9.71 268	21	9.77 992	28	0.22 008	9.93 276	7	56
5	9.71 289	21	9.78 020	29	0.21 980	9.93 269	8	55
6	9.71 310	21	9.78 049	28	0.21 951	9.93 261	8	54
7	9.71 331	21	9.78 077	29	0.21 923	9.93 253	7	53
8	9.71 352	21	9.78 106	29	0.21 894	9.93 246	8	52
9	9.71 373	20	9.78 135̄	28	0.21 865	9.93 238	8	51
10	9.71 393	21	9.78 163	29	0.21 837	9.93 230	7	50
11	9.71 414	21	9.78 192	28	0.21 808	9.93 223	8	49
12	9.71 435	21	9.78 220	29	0.21 780	9.93 215	8	48
13	9.71 456	21	9.78 249	28	0.21 751	9.93 207	7	47
14	9.71 477	21	9.78 277	29	0.21 723	9.93 200	8	46
15	9.71 498	21	9.78 306	28	0.21 694	9.93 192	8	45
16	9.71 519	20	9.78 334	29	0.21 666	9.93 184	7	44
17	9.71 539	21	9.78 363	28	0.21 637	9.93 177	8	43
18	9.71 560	21	9.78 391	28	0.21 609	9.93 169	8	42
19	9.71 581	21	9.78 419	29	0.21 581	9.93 161	7	41
20	9.71 602	20	9.78 448	28	0.21 552	9.93 154	8	40
21	9.71 622	21	9.78 476	29	0.21 524	9.93 146	8	39
22	9.71 643	21	9.78 505̄	28	0.21 495	9.93 138	7	38
23	9.71 664	21	9.78 533	29	0.21 467	9.93 131	8	37
24	9.71 685̄	20	9.78 562	28	0.21 438	9.93 123	8	36
25	9.71 705	21	9.78 590	28	0.21 410	9.93 115	7	35
26	9.71 726	21	9.78 618	29	0.21 382	9.93 108	8	34
27	9.71 747	20	9.78 647	28	0.21 353̱	9.93 100	8	33
28	9.71 767	21	9.78 675	29	0.21 325̄	9.93 092	8	32
29	9.71 788	21	9.78 704	28	0.21 296	9.93 084	7	31
30	9.71 809	20	9.78 732	28	0.21 268	9.93 077	8	30
31	9.71 829	21	9.78 760	29	0.21 240	9.93 069	8	29
32	9.71 850	20	9.78 789	28	0.21 211	9.93 061	8	28
33	9.71 870	21	9.78 817	28	0.21 183̱	9.93 053	7	27
34	9.71 891	20	9.78 845	29	0.21 155̄	9.93 046	8	26
35	9.71 911	21	9.78 874	28	0.21 126	9.93 038	8	25
36	9.71 932	20	9.78 902	28	0.21 098	9.93 030	8	24
37	9.71 952	21	9.78 930	29	0.21 070	9.93 022	8	23
38	9.71 973	21	9.78 959	28	0.21 041	9.93 014	7	22
39	9.71 994	20	9.78 987	28	0.21 013	9.93 007	8	21
40	9.72 014	20	9.79 015	28	0.20 985̄	9.92 999	8	20
41	9.72 034	21	9.79 043	29	0.20 957	9.92 991	8	19
42	9.72 055̄	20	9.79 072	28	0.20 928	9.92 983	7	18
43	9.72 075	21	9.79 100	28	0.20 900	9.92 976	8	17
44	9.72 096	20	9.79 128	28	0.20 872	9.92 968	8	16
45	9.72 116	21	9.79 156	29	0.20 844	9.92 960	8	15
46	9.72 137	20	9.79 185̄	28	0.20 815	9.92 952	8	14
47	9.72 157	20	9.79 213	28	0.20 787	9.92 944	8	13
48	9.72 177	21	9.79 241	28	0.20 759	9.92 936	7	12
49	9.72 198	20	9.79 269	28	0.20 731	9.92 929	8	11
50	9.72 218	20	9.79 297	29	0.20 703	9.92 921	8	10
51	9.72 238	21	9.79 326	28	0.20 674	9.92 913	8	9
52	9.72 259	20	9.79 354	28	0.20 646	9.92 905	8	8
53	9.72 279	20	9.79 382	28	0.20 618	9.92 897	8	7
54	9.72 299	21	9.79 410	28	0.20 590	9.92 889	8	6
55	9.72 320	20	9.79 438	28	0.20 562	9.92 881	7	5
56	9.72 340	20	9.79 466	29	0.20 534	9.92 874	8	4
57	9.72 360	21	9.79 495̄	28	0.20 505	9.92 866	8	3
58	9.72 381	20	9.79 523	28	0.20 477	9.92 858	8	2
59	9.72 401	20	9.79 551	28	0.20 449	9.92 850	8	1
60	9.72 421		9.79 579		0.20 421	9.92 842		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	29	28	21	20	8	7
.1	2.9	2.8	2.1	2.0	0.8	0.7
.2	5.8	5.6	4.2	4.0	1.6	1.4
.3	8.7	8.4	6.3	6.0	2.4	2.1
.4	11.6	11.2	8.4	8.0	3.2	2.8
.5	14.5	14.0	10.5	10.0	4.0	3.5
.6	17.4	16.8	12.6	12.0	4.8	4.2
.7	20.3	19.6	14.7	14.0	5.6	4.9
.8	23.2	22.4	16.8	16.0	6.4	5.6
.9	26.1	25.2	18.9	18.0	7.2	6.3

Prop. Pts.

58°

32°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.72 421	20	9.79 579	28	0.20 421	9.92 842	8	60
1	9.72 441	20	9.79 607	28	0.20 393	9.92 834	8	59
2	9.72 461	21	9.79 635	28	0.20 365	9.92 826	8	58
3	9.72 482	20	9.79 663	28	0.20 337	9.92 818	8	57
4	9.72 502	20	9.79 691	28	0.20 309	9.92 810	7	56
5	9.72 522	20	9.79 719	28	0.20 281	9.92 803	8	55
6	9.72 542	20	9.79 747	29	0.20 253	9.92 795	8	54
7	9.72 562	20	9.79 776	28	0.20 224	9.92 787	8	53
8	9.72 582	20	9.79 804	28	0.20 196	9.92 779	8	52
9	9.72 602	20	9.79 832	28	0.20 168	9.92 771	8	51
10	9.72 622	21	9.79 860	28	0.20 140	9.92 763	8	50
11	9.72 643	20	9.79 888	28	0.20 112	9.92 755	8	49
12	9.72 663	20	9.79 916	28	0.20 084	9.92 747	8	48
13	9.72 683	20	9.79 944	28	0.20 056	9.92 739	8	47
14	9.72 703	20	9.79 972	28	0.20 028	9.92 731	8	46
15	9.72 723	20	9.80 000	28	0.20 000	9.92 723	8	45
16	9.72 743	20	9.80 028	28	0.19 972	9.92 715	8	44
17	9.72 763	20	9.80 056	28	0.19 944	9.92 707	8	43
18	9.72 783	20	9.80 084	28	0.19 916	9.92 699	8	42
19	9.72 803	20	9.80 112	28	0.19 888	9.92 691	8	41
20	9.72 823	20	9.80 140	28	0.19 860	9.92 683	8	40
21	9.72 843	20	9.80 168	27	0.19 832	9.92 675	8	39
22	9.72 863	20	9.80 195	28	0.19 805	9.92 667	8	38
23	9.72 883	19	9.80 223	28	0.19 777	9.92 659	8	37
24	9.72 902	20	9.80 251	28	0.19 749	9.92 651	8	36
25	9.72 922	20	9.80 279	28	0.19 721	9.92 643	8	35
26	9.72 942	20	9.80 307	28	0.19 693	9.92 635	8	34
27	9.72 962	20	9.80 335	28	0.19 665	9.92 627	8	33
28	9.72 982	20	9.80 363	28	0.19 637	9.92 619	8	32
29	9.73 002	20	9.80 391	28	0.19 609	9.92 611	8	31
30	9.73 022	19	9.80 419	28	0.19 581	9.92 603	8	30
31	9.73 041	20	9.80 447	27	0.19 553	9.92 595	8	29
32	9.73 061	20	9.80 474	28	0.19 526	9.92 587	8	28
33	9.73 081	20	9.80 502	28	0.19 498	9.92 579	8	27
34	9.73 101	20	9.80 530	28	0.19 470	9.92 571	8	26
35	9.73 121	19	9.80 558	28	0.19 442	9.92 563	8	25
36	9.73 140	20	9.80 586	28	0.19 414	9.92 555	9	24
37	9.73 160	20	9.80 614	28	0.19 386	9.92 546	8	23
38	9.73 180	20	9.80 642	27	0.19 358	9.92 538	8	22
39	9.73 200	19	9.80 669	28	0.19 331	9.92 530	8	21
40	9.73 219	20	9.80 697	28	0.19 303	9.92 522	8	20
41	9.73 239	20	9.80 725	28	0.19 275	9.92 514	8	19
42	9.73 259	19	9.80 753	28	0.19 247	9.92 506	8	18
43	9.73 278	20	9.80 781	27	0.19 219	9.92 498	8	17
44	9.73 298	20	9.80 808	28	0.19 192	9.92 490	8	16
45	9.73 318	19	9.80 836	28	0.19 164	9.92 482	9	15
46	9.73 337	20	9.80 864	28	0.19 136	9.92 473	8	14
47	9.73 357	20	9.80 892	27	0.19 108	9.92 465	8	13
48	9.73 377	19	9.80 919	28	0.19 081	9.92 457	8	12
49	9.73 396	20	9.80 947	28	0.19 053	9.92 449	8	11
50	9.73 416	19	9.80 975	28	0.19 025	9.92 441	8	10
51	9.73 435	20	9.81 003	27	0.18 997	9.92 433	8	9
52	9.73 455	19	9.81 030	28	0.18 970	9.92 425	9	8
53	9.73 474	20	9.81 058	28	0.18 942	9.92 416	8	7
54	9.73 494	19	9.81 086	27	0.18 914	9.92 408	8	6
55	9.73 513	20	9.81 113	28	0.18 887	9.92 400	8	5
56	9.73 533	19	9.81 141	28	0.18 859	9.92 392	8	4
57	9.73 552	20	9.81 169	27	0.18 831	9.92 384	8	3
58	9.73 572	19	9.81 196	28	0.18 804	9.92 376	9	2
59	9.73 591	20	9.81 224	28	0.18 776	9.92 367	8	1
60	9.73 611		9.81 252		0.18 748	9.92 359		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	29	28
.1	2.9	2.8
.2	5.8	5.6
.3	8.7	8.4
.4	11.6	11.2
.5	14.5	14.0
.6	17.4	16.8
.7	20.3	19.6
.8	23.2	22.4
.9	26.1	25.2

	27
.1	2.7
.2	5.4
.3	8.1
.4	10.8
.5	13.5
.6	16.2
.7	18.9
.8	21.6
.9	24.3

	21	20
.1	2.1	2.0
.2	4.2	4.0
.3	6.3	6.0
.4	8.4	8.0
.5	10.5	10.0
.6	12.6	12.0
.7	14.7	14.0
.8	16.8	16.0
.9	18.9	18.0

	19	9
.1	1.9	0.9
.2	3.8	1.8
.3	5.7	2.7
.4	7.6	3.6
.5	9.5	4.5
.6	11.4	5.4
.7	13.3	6.3
.8	15.2	7.2
.9	17.1	8.1

	8	7
.1	0.8	0.7
.2	1.6	1.4
.3	2.4	2.1
.4	3.2	2.8
.5	4.0	3.5
.6	4.8	4.2
.7	5.6	4.9
.8	6.4	5.6
.9	7.2	6.3

Prop. Pts.

57°

33°

'	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.73 611	19	9.81 252	27	0.18 748	9.92 359	8	60
1	9.73 630	20	9.81 279	28	0.18 721	9.92 351	8	59
2	9.73 650	19	9.81 307	28	0.18 693	9.92 343	8	58
3	9.73 669	20	9.81 335	27	0.18 665	9.92 335	9	57
4	9.73 689	19	9.81 362	28	0.18 638	9.92 326	8	56
5	9.73 708	19	9.81 390	28	0.18 610	9.92 318	8	55
6	9.73 727	20	9.81 418	27	0.18 582	9.92 310	8	54
7	9.73 747	19	9.81 445	28	0.18 555	9.92 302	9	53
8	9.73 766	19	9.81 473	27	0.18 527	9.92 293	8	52
9	9.73 785	20	9.81 500	28	0.18 500	9.92 285	8	51
10	9.73 805	19	9.81 528	28	0.18 472	9.92 277	8	50
11	9.73 824	19	9.81 556	27	0.18 444	9.92 269	9	49
12	9.73 843	20	9.81 583	28	0.18 417	9.92 260	8	48
13	9.73 863	19	9.81 611	27	0.18 389	9.92 252	8	47
14	9.73 882	19	9.81 638	28	0.18 362	9.92 244	9	46
15	9.73 901	20	9.81 666	27	0.18 334	9.92 235	8	45
16	9.73 921	19	9.81 693	28	0.18 307	9.92 227	8	44
17	9.73 940	19	9.81 721	27	0.18 279	9.92 219	8	43
18	9.73 959	19	9.81 748	28	0.18 252	9.92 211	9	42
19	9.73 978	19	9.81 776	27	0.18 224	9.92 202	8	41
20	9.73 997	20	9.81 803	28	0.18 197	9.92 194	8	40
21	9.74 017	19	9.81 831	27	0.18 169	9.92 186	9	39
22	9.74 036	19	9.81 858	28	0.18 142	9.92 177	8	38
23	9.74 055	19	9.81 886	27	0.18 114	9.92 169	8	37
24	9.74 074	19	9.81 913	28	0.18 087	9.92 161	9	36
25	9.74 093	20	9.81 941	27	0.18 059	9.92 152	8	35
26	9.74 113	19	9.81 968	28	0.18 032	9.92 144	8	34
27	9.74 132	19	9.81 996	27	0.18 004	9.92 136	9	33
28	9.74 151	19	9.82 023	28	0.17 977	9.92 127	8	32
29	9.74 170	19	9.82 051	27	0.17 949	9.92 119	8	31
30	9.74 189	19	9.82 078	28	0.17 922	9.92 111	9	30
31	9.74 208	19	9.82 106	27	0.17 894	9.92 102	8	29
32	9.74 227	19	9.82 133	28	0.17 867	9.92 094	8	28
33	9.74 246	19	9.82 161	27	0.17 839	9.92 086	9	27
34	9.74 265	19	9.82 188	27	0.17 812	9.92 077	8	26
35	9.74 284	19	9.82 215	28	0.17 785	9.92 069	9	25
36	9.74 303	19	9.82 243	27	0.17 757	9.92 060	8	24
37	9.74 322	19	9.82 270	28	0.17 730	9.92 052	8	23
38	9.74 341	19	9.82 298	27	0.17 702	9.92 044	9	22
39	9.74 360	19	9.82 325	27	0.17 675	9.92 035	8	21
40	9.74 379	19	9.82 352	28	0.17 648	9.92 027	9	20
41	9.74 398	19	9.82 380	27	0.17 620	9.92 018	8	19
42	9.74 417	19	9.82 407	28	0.17 593	9.92 010	8	18
43	9.74 436	19	9.82 435	27	0.17 565	9.92 002	9	17
44	9.74 455	19	9.82 462	27	0.17 538	9.91 993	8	16
45	9.74 474	19	9.82 489	28	0.17 511	9.91 985	9	15
46	9.74 493	19	9.82 517	27	0.17 483	9.91 976	8	14
47	9.74 512	19	9.82 544	27	0.17 456	9.91 968	9	13
48	9.74 531	18	9.82 571	28	0.17 429	9.91 959	8	12
49	9.74 549	19	9.82 599	27	0.17 401	9.91 951	9	11
50	9.74 568	19	9.82 626	27	0.17 374	9.91 942	8	10
51	9.74 587	19	9.82 653	28	0.17 347	9.91 934	9	9
52	9.74 606	19	9.82 681	27	0.17 319	9.91 925	8	8
53	9.74 625	19	9.82 708	27	0.17 292	9.91 917	9	7
54	9.74 644	18	9.82 735	27	0.17 265	9.91 908	8	6
55	9.74 662	19	9.82 762	28	0.17 238	9.91 900	9	5
56	9.74 681	19	9.82 790	27	0.17 210	9.91 891	8	4
57	9.74 700	19	9.82 817	27	0.17 183	9.91 883	9	3
58	9.74 719	18	9.82 844	27	0.17 156	9.91 874	8	2
59	9.74 737	19	9.82 871	28	0.17 129	9.91 866	9	1
60	9.74 756		9.82 899		0.17 101	9.91 857		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	'

Prop. Pts.

	28	27
.1	2.8	2.7
.2	5.6	5.4
.3	8.4	8.1
.4	11.2	10.8
.5	14.0	13.5
.6	16.8	16.2
.7	19.6	18.9
.8	22.4	21.6
.9	25.2	24.3

	20
.1	2.0
.2	4.0
.3	6.0
.4	8.0
.5	10.0
.6	12.0
.7	14.0
.8	16.0
.9	18.0

	19
.1	1.9
.2	3.8
.3	5.7
.4	7.6
.5	9.5
.6	11.4
.7	13.3
.8	15.2
.9	17.1

	18
.1	1.8
2	3.6
.3	5.4
.4	7.2
.5	9.0
.6	10.8
.7	12.6
.8	14.4
.9	16.2

	9	8
.1	0.9	0.8
.2	1.8	1.6
.3	2.7	2.4
.4	3.6	3.2
.5	4.5	4.0
.6	5.4	4.8
.7	6.3	5.6
.8	7.2	6.4
.9	8.1	7.2

Prop. Pts.

56°

34°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.74 756	19	9.82 899	27	0.17 101	9.91 857	8	60
1	9.74 775	19	9.82 926	27	0.17 074	9.91 849	9	59
2	9.74 794	18	9.82 953	27	0.17 047	9.91 840	8	58
3	9.74 812	19	9.82 980	28	0.17 020	9.91 832	9	57
4	9.74 831	19	9.83 008	27	0.16 992	9.91 823	8	56
5	9.74 850	18	9.83 035	27	0.16 965	9.91 815	9	55
6	9.74 868	19	9.83 062	27	0.16 938	9.91 806	8	54
7	9.74 887	19	9.83 089	28	0.16 911	9.91 798	9	53
8	9.74 906	18	9.83 117	27	0.16 883	9.91 789	8	52
9	9.74 924	19	9.83 144	27	0.16 856	9.91 781	9	51
10	9.74 943	18	9.83 171	27	0.16 829	9.91 772	9	50
11	9.74 961	19	9.83 198	27	0.16 802	9.91 763	8	49
12	9.74 980	19	9.83 225	27	0.16 775	9.91 755	9	48
13	9.74 999	18	9.83 252	28	0.16 748	9.91 746	8	47
14	9.75 017	19	9.83 280	27	0.16 720	9.91 738	9	46
15	9.75 036	18	9.83 307	27	0.16 693	9.91 729	9	45
16	9.75 054	19	9.83 334	27	0.16 666	9.91 720	8	44
17	9.75 073	18	9.83 361	27	0.16 639	9.91 712	9	43
18	9.75 091	19	9.83 388	27	0.16 612	9.91 703	8	42
19	9.75 110	18	9.83 415	27	0.16 585	9.91 695	9	41
20	9.75 128	19	9.83 442	28	0.16 558	9.91 686	9	40
21	9.75 147	18	9.83 470	27	0.16 530	9.91 677	8	39
22	9.75 165	19	9.83 497	27	0.16 503	9.91 669	9	38
23	9.75 184	18	9.83 524	27	0.16 476	9.91 660	9	37
24	9.75 202	19	9.83 551	27	0.16 449	9.91 651	8	36
25	9.75 221	18	9.83 578	27	0.16 422	9.91 643	9	35
26	9.75 239	19	9.83 605	27	0.16 395	9.91 634	9	34
27	9.75 258	18	9.83 632	27	0.16 368	9.91 625	8	33
28	9.75 276	18	9.83 659	27	0.16 341	9.91 617	9	32
29	9.75 294	19	9.83 686	27	0.16 314	9.91 608	9	31
30	9.75 313	18	9.83 713	27	0.16 287	9.91 599	8	30
31	9.75 331	19	9.83 740	28	0.16 260	9.91 591	9	29
32	9.75 350	18	9.83 768	27	0.16 232	9.91 582	9	28
33	9.75 368	18	9.83 795	27	0.16 205	9.91 573	8	27
34	9.75 386	19	9.83 822	27	0.16 178	9.91 565	9	26
35	9.75 405	18	9.83 849	27	0.16 151	9.91 556	9	25
36	9.75 423	18	9.83 876	27	0.16 124	9.91 547	9	24
37	9.75 441	18	9.83 903	27	0.16 097	9.91 538	8	23
38	9.75 459	19	9.83 930	27	0.16 070	9.91 530	9	22
39	9.75 478	18	9.83 957	27	0.16 043	9.91 521	9	21
40	9.75 496	18	9.83 984	27	0.16 016	9.91 512	8	20
41	9.75 514	19	9.84 011	27	0.15 989	9.91 504	9	19
42	9.75 533	18	9.84 038	27	0.15 962	9.91 495	9	18
43	9.75 551	18	9.84 065	27	0.15 935	9.91 486	9	17
44	9.75 569	18	9.84 092	27	0.15 908	9.91 477	8	16
45	9.75 587	18	9.84 119	27	0.15 881	9.91 469	9	15
46	9.75 605	19	9.84 146	27	0.15 854	9.91 460	9	14
47	9.75 624	18	9.84 173	27	0.15 827	9.91 451	9	13
48	9.75 642	18	9.84 200	27	0.15 800	9.91 442	9	12
49	9.75 660	18	9.84 227	27	0.15 773	9.91 433	8	11
50	9.75 678	18	9.84 254	26	0.15 746	9.91 425	9	10
51	9.75 696	18	9.84 280	27	0.15 720	9.91 416	9	9
52	9.75 714	19	9.84 307	27	0.15 693	9.91 407	9	8
53	9.75 733	18	9.84 334	27	0.15 666	9.91 398	9	7
54	9.75 751	18	9.84 361	27	0.15 639	9.91 389	8	6
55	9.75 769	18	9.84 388	27	0.15 612	9.91 381	9	5
56	9.75 787	18	9.84 415	27	0.15 585	9.91 372	9	4
57	9.75 805	18	9.84 442	27	0.15 558	9.91 363	9	3
58	9.75 823	18	9.84 469	27	0.15 531	9.91 354	9	2
59	9.75 841	18	9.84 496	27	0.15 504	9.91 345	9	1
60	9.75 859		9.84 523		0.15 477	9.91 336		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	28	27
.1	2.8	2.7
.2	5.6	5.4
.3	8.4	8.1
.4	11.2	10.8
.5	14.0	13.5
.6	16.8	16.2
.7	19.6	18.9
.8	22.4	21.6
.9	25.2	24.3

	26
.1	2.6
.2	5.2
.3	7.8
.4	10.4
.5	13.0
.6	15.6
.7	18.2
.8	20.8
.9	23.4

	19
.1	1.9
.2	3.8
.3	5.7
.4	7.6
.5	9.5
.6	11.4
.7	13.3
.8	15.2
.9	17.1

	18
.1	1.8
.2	3.6
.3	5.4
.4	7.2
.5	9.0
.6	10.8
.7	12.6
.8	14.4
.9	16.2

	9	8
.1	0.9	0.8
.2	1.8	1.6
.3	2.7	2.4
.4	3.6	3.2
.5	4.5	4.0
.6	5.4	4.8
.7	6.3	5.6
.8	7.2	6.4
.9	8.1	7.2

Prop. Pts.

55°

35°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.75 859	18	9.84 523	27	0.15 477	9.91 336	8	60
1	9.75 877	18	9.84 550	26	0.15 450	9.91 328	9	59
2	9.75 895	18	9.84 576	27	0.15 424	9.91 319	9	58
3	9.75 913	18	9.84 603	27	0.15 397	9.91 310	9	57
4	9.75 931	18	9.84 630	27	0.15 370	9.91 301	9	56
5	9.75 949	18	9.84 657	27	0.15 343	9.91 292	9	55
6	9.75 967	18	9.84 684	27	0.15 316	9.91 283	9	54
7	9.75 985	18	9.84 711	27	0.15 289	9.91 274	8	53
8	9.76 003	18	9.84 738	26	0.15 262	9.91 266	9	52
9	9.76 021	18	9.84 764	27	0.15 236	9.91 257	9	51
10	9.76 039	18	9.84 791	27	0.15 209	9.91 248	9	50
11	9.76 057	18	9.84 818	27	0.15 182	9.91 239	9	49
12	9.76 075	18	9.84 845	27	0.15 155	9.91 230	9	48
13	9.76 093	18	9.84 872	27	0.15 128	9.91 221	9	47
14	9.76 111	18	9.84 899	26	0.15 101	9.91 212	9	46
15	9.76 129	17	9.84 925	27	0.15 075	9.91 203	9	45
16	9.76 146	18	9.84 952	27	0.15 048	9.91 194	9	44
17	9.76 164	18	9.84 979	27	0.15 021	9.91 185	9	43
18	9.76 182	18	9.85 006	27	0.14 994	9.91 176	9	42
19	9.76 200	18	9.85 033	26	0.14 967	9.91 167	9	41
20	9.76 218	18	9.85 059	27	0.14 941	9.91 158	9	40
21	9.76 236	17	9.85 086	27	0.14 914	9.91 149	8	39
22	9.76 253	18	9.85 113	27	0.14 887	9.91 141	9	38
23	9.76 271	18	9.85 140	26	0.14 860	9.91 132	9	37
24	9.76 289	18	9.85 166	27	0.14 834	9.91 123	9	36
25	9.76 307	17	9.85 193	27	0.14 807	9.91 114	9	35
26	9.76 324	18	9.85 220	27	0.14 780	9.91 105	9	34
27	9.76 342	18	9.85 247	26	0.14 753	9 91 096	9	33
28	9.76 360	18	9.85 273	27	0.14 727	9.91 087	9	32
29	9.76 378	17	9.85 300	27	0.14 700	9.91 078	9	31
30	9.76 395	18	9.85 327	27	0.14 673	9.91 069	9	30
31	9.76 413	18	9.85 354	26	0.14 646	9.91 060	9	29
32	9.76 431	17	9.85 380	27	0.14 620	9.91 051	9	28
33	9.76 448	18	9.85 407	27	0.14 593	9.91 042	9	27
34	9.76 466	18	9.85 434	26	0.14 566	9.91 033	10	26
35	9.76 484	17	9.85 460	27	0.14 540	9.91 023	9	25
36	9.76 501	18	9.85 487	27	0.14 513	9.91 014	9	24
37	9.76 519	18	9.85 514	26	0.14 486	9.91 005	9	23
38	9.76 537	17	9.85 540	27	0.14 460	9.90 996	9	22
39	9.76 554	18	9.85 567	27	0.14 433	9.90 987	9	21
40	9.76 572	18	9.85 594	26	0.14 406	9.90 978	9	20
41	9.76 590	17	9.85 620	27	0.14 380	9.90 969	9	19
42	9.76 607	18	9.85 647	27	0.14 353	9.90 960	9	18
43	9.76 625	17	9.85 674	26	0.14 326	9.90 951	9	17
44	9.76 642	18	9.85 700	27	0.14 300	9.90 942	9	16
45	9.76 660	17	9.85 727	27	0.14 273	9.90 933	9	15
46	9.76 677	18	9.85 754	26	0.14 246	9.90 924	9	14
47	9.76 695	17	9.85 780	27	0.14 220	9.90 915	9	13
48	9.76 712	18	9.85 807	27	0.14 193	9.90 906	10	12
49	9.76 730	17	9.85 834	26	0.14 166	9.90 896	9	11
50	9.76 747	18	9.85 860	27	0.14 140	9.90 887	9	10
51	9.76 765	17	9.85 887	26	0.14 113	9.90 878	9	9
52	9.76 782	18	9.85 913	27	0.14 087	9.90 869	9	8
53	9.76 800	17	9.85 940	27	0.14 060	9.90 860	9	7
54	9.76 817	18	9.85 967	26	0.14 033	9.90 851	9	6
55	9.76 835	17	9.85 993	27	0.14 007	9.90 842	10	5
56	9.76 852	18	9.86 020	26	0.13 980	9.90 832	9	4
57	9.76 870	17	9.86 046	27	0.13 954	9.90 823	9	3
58	9.76 887	17	9.86 073	27	0.13 927	9.90 814	9	2
59	9.76 904	18	9.86 100	26	0.13 900	9.90 805	9	1
60	9.76 922		9.86 126		0.13 874	9.90 796		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	27	26
.1	2.7	2.6
.2	5.4	5.2
.3	8.1	7.8
.4	10.8	10.4
.5	13.5	13.0
.6	16.2	15.6
.7	18.9	18.2
.8	21.6	20.8
.9	24.3	23.4

	18
.1	1.8
.2	3.6
.3	5.4
.4	7.2
.5	9.0
.6	10.8
.7	12.6
.8	14.4
.9	16.2

	17
.1	1.7
.2	3.4
.3	5.1
.4	6.8
.5	8.5
.6	10.2
.7	11.9
.8	13.6
.9	15.3

	10
.1	1.0
.2	2.0
.3	3.0
.4	4.0
.5	5.0
.6	6.0
.7	7.0
.8	8.0
.9	9.0

	9	8
.1	0.9	0.8
.2	1.8	1.6
.3	2.7	2.4
.4	3.6	3.2
.5	4.5	4.0
.6	5.4	4.8
.7	6.3	5.6
.8	7.2	6.4
.9	8.1	7.2

Prop. Pts.

54°

36°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.76 922	17	9.86 126	27	0.13 874	9.90 796	9	60
1	9.76 939	18	9.86 153	26	0.13 847	9.90 787	10	59
2	9.76 957	17	9.86 179	27	0.13 821	9.90 777	9	58
3	9.76 974	17	9.86 206	26	0.13 794	9.90 768	9	57
4	9.76 991	18	9.86 232	27	0.13 768	9.90 759	9	56
5	9.77 009	17	9.86 259	26	0.13 741	9.90 750	9	55
6	9.77 026	17	9.86 285	27	0.13 715	9.90 741	10	54
7	9.77 043	18	9.86 312	26	0.13 688	9.90 731	9	53
8	9.77 061	17	9.86 338	27	0.13 662	9.90 722	9	52
9	9.77 078	17	9.86 365	27	0.13 635	9.90 713	9	51
10	9.77 095	17	9.86 392	26	0.13 608	9.90 704	10	50
11	9.77 112	18	9.86 418	27	0.13 582	9.90 694	9	49
12	9.77 130	17	9.86 445	26	0.13 555	9.90 685	9	48
13	9.77 147	17	9.86 471	27	0.13 529	9.90 676	9	47
14	9.77 164	17	9.86 498	26	0.13 502	9.90 667	10	46
15	9.77 181	18	9.86 524	27	0.13 476	9.90 657	9	45
16	9.77 199	17	9.86 551	26	0.13 449	9.90 648	9	44
17	9.77 216	17	9.86 577	26	0.13 423	9.90 639	9	43
18	9.77 233	17	9.86 603	27	0.13 397	9.90 630	10	42
19	9.77 250	18	9.86 630	26	0.13 370	9.90 620	9	41
20	9.77 268	17	9.86 656	27	0.13 344	9.90 611	9	40
21	9.77 285	17	9.86 683	26	0.13 317	9.90 602	10	39
22	9.77 302	17	9.86 709	27	0.13 291	9.90 592	9	38
23	9.77 319	17	9.86 736	26	0.13 264	9.90 583	9	37
24	9.77 336	17	9.86 762	27	0.13 238	9.90 574	9	36
25	9.77 353	17	9.86 789	26	0.13 211	9.90 565	10	35
26	9.77 370	17	9.86 815	27	0.13 185	9.90 555	9	34
27	9.77 387	18	9.86 842	26	0.13 158	9.90 546	9	33
28	9.77 405	17	9.86 868	26	0.13 132	9.90 537	10	32
29	9.77 422	17	9.86 894	27	0.13 106	9.90 527	9	31
30	9.77 439	17	9.86 921	26	0.13 079	9.90 518	9	30
31	9.77 456	17	9.86 947	27	0.13 053	9.90 509	10	29
32	9.77 473	17	9.86 974	26	0.13 026	9.90 499	9	28
33	9.77 490	17	9.87 000	27	0.13 000	9.90 490	10	27
34	9.77 507	17	9.87 027	26	0.12 973	9.90 480	9	26
35	9.77 524	17	9.87 053	26	0.12 947	9.90 471	9	25
36	9.77 541	17	9.87 079	27	0.12 921	9.90 462	10	24
37	9.77 558	17	9.87 106	26	0.12 894	9.90 452	9	23
38	9.77 575	17	9.87 132	26	0.12 868	9.90 443	9	22
39	9.77 592	17	9.87 158	27	0.12 842	9.90 434	10	21
40	9.77 609	17	9.87 185	26	0.12 815	9.90 424	9	20
41	9.77 626	17	9.87 211	27	0.12 789	9.90 415	10	19
42	9.77 643	17	9.87 238	26	0.12 762	9.90 405	9	18
43	9.77 660	17	9.87 264	26	0.12 736	9.90 396	10	17
44	9.77 677	17	9.87 290	27	0.12 710	9.90 386	9	16
45	9.77 694	17	9.87 317	26	0.12 683	9.90 377	9	15
46	9.77 711	17	9.87 343	26	0.12 657	9.90 368	10	14
47	9.77 728	16	9.87 369	27	0.12 631	9.90 358	9	13
48	9.77 744	17	9.87 396	26	0.12 604	9.90 349	10	12
49	9.77 761	17	9.87 422	26	0.12 578	9.90 339	9	11
50	9.77 778	17	9.87 448	27	0.12 552	9.90 330	10	10
51	9.77 795	17	9.87 475	26	0.12 525	9.90 320	9	9
52	9.77 812	17	9.87 501	26	0.12 499	9.90 311	10	8
53	9.77 829	17	9.87 527	27	0.12 473	9.90 301	9	7
54	9.77 846	16	9.87 554	26	0.12 446	9.90 292	10	6
55	9.77 862	17	9.87 580	26	0.12 420	9.90 282	9	5
56	9.77 879	17	9.87 606	27	0.12 394	9.90 273	10	4
57	9.77 896	17	9.87 633	26	0.12 367	9.90 263	9	3
58	9.77 913	17	9.87 659	26	0.12 341	9.90 254	10	2
59	9.77 930	16	9.87 685	26	0.12 315	9.90 244	9	1
60	9.77 946		9.87 711		0.12 289	9.90 235		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	27	26
.1	2.7	2.6
.2	5.4	5.2
.3	8.1	7.8
.4	10.8	10.4
.5	13.5	13.0
.6	16.2	15.6
.7	18.9	18.2
.8	21.6	20.8
.9	24.3	23.4

	18
.1	1.8
.2	3.6
.3	5.4
.4	7.2
.5	9.0
.6	10.8
.7	12.6
.8	14.4
.9	16.2

	17
.1	1.7
.2	3.4
.3	5.1
.4	6.8
.5	8.5
.6	10.2
.7	11.9
.8	13.6
.9	15.3

	16
.1	1.6
.2	3.2
.3	4.8
.4	6.4
.5	8.0
.6	9.6
.7	11.2
.8	12.8
.9	14.4

	10	9
.1	1.0	0.9
.2	2.0	1.8
.3	3.0	2.7
.4	4.0	3.6
.5	5.0	4.5
.6	6.0	5.4
.7	7.0	6.3
.8	8.0	7.2
.9	9.0	8.1

Prop. Pts.

53°

37°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.77 946	17	9.87 711	27	0.12 289	9.90 235	10	**60**
1	9.77 963	17	9.87 738	26	0.12 262	9.90 225	9	59
2	9.77 980	17	9.87 764	26	0.12 236	9.90 216	10	58
3	9.77 997	16	9.87 790	27	0.12 210	9.90 206	9	57
4	9.78 013	17	9.87 817	26	0.12 183	9.90 197	10	56
5	9.78 030	17	9.87 843	26	0.12 157	9.90 187	9	55
6	9.78 047	16	9.87 869	26	0.12 131	9.90 178	10	54
7	9.78 063	17	9.87 895	27	0.12 105	9.90 168	9	53
8	9.78 080	17	9.87 922	26	0.12 078	9.90 159	10	52
9	9.78 097	16	9.87 948	26	0.12 052	9.90 149	10	51
10	9.78 113	17	9.87 974	26	0.12 026	9.90 139	9	**50**
11	9.78 130	17	9.88 000	27	0.12 000	9.90 130	10	49
12	9.78 147	16	9.88 027	26	0.11 973	9.90 120	9	48
13	9.78 163	17	9.88 053	26	0.11 947	9.90 111	10	47
14	9.78 180	17	9.88 079	26	0.11 921	9.90 101	10	46
15	9.78 197	16	9.88 105	26	0.11 895	9.90 091	9	45
16	9.78 213	17	9.88 131	27	0.11 869	9.90 082	10	44
17	9.78 230	16	9.88 158	26	0.11 842	9.90 072	9	43
18	9.78 246	17	9.88 184	26	0.11 816	9.90 063	10	42
19	9.78 263	17	9.88 210	26	0.11 790	9.90 053	10	41
20	9.78 280	16	9.88 236	26	0.11 764	9.90 043	9	**40**
21	9.78 296	17	9.88 262	27	0.11 738	9.90 034	10	39
22	9.78 313	16	9.88 289	26	0.11 711	9.90 024	10	38
23	9.78 329	17	9.88 315	26	0.11 685	9.90 014	9	37
24	9.78 346	16	9.88 341	26	0.11 659	9.90 005	10	36
25	9.78 362	17	9.88 367	26	0.11 633	9.89 995	10	35
26	9.78 379	16	9.88 393	27	0.11 607	9.89 985	9	34
27	9.78 395	17	9.88 420	26	0.11 580	9.89 976	10	33
28	9.78 412	16	9.88 446	26	0.11 554	9.89 966	10	32
29	9.78 428	17	9.88 472	26	0.11 528	9.89 956	9	31
30	9.78 445	16	9.88 498	26	0.11 502	9.89 947	10	**30**
31	9.78 461	17	9.88 524	26	0.11 476	9.89 937	10	29
32	9.78 478	16	9.88 550	27	0.11 450	9.89 927	9	28
33	9.78 494	16	9.88 577	26	0.11 423	9.89 918	10	27
34	9.78 510	17	9.88 603	26	0.11 397	9.89 908	10	26
35	9.78 527	16	9.88 629	26	0.11 371	9.89 898	10	25
36	9.78 543	17	9.88 655	26	0.11 345	9.89 888	9	24
37	9.78 560	16	9.88 681	26	0.11 319	9.89 879	10	23
38	9.78 576	16	9.88 707	26	0.11 293	9.89 869	10	22
39	9.78 592	17	9.88 733	26	0.11 267	9.89 859	10	21
40	9.78 609	16	9.88 759	27	0.11 241	9.89 849	9	**20**
41	9.78 625	17	9.88 786	26	0.11 214	9.89 840	10	19
42	9.78 642	16	9.88 812	26	0.11 188	9.89 830	10	18
43	9.78 658	16	9.88 838	26	0.11 162	9.89 820	10	17
44	9.78 674	17	9.88 864	26	0.11 136	9.89 810	9	16
45	9.78 691	16	9.88 890	26	0.11 110	9.89 801	10	15
46	9.78 707	16	9.88 916	26	0.11 084	9.89 791	10	14
47	9.78 723	16	9.88 942	26	0.11 058	9.89 781	10	13
48	9.78 739	17	9.88 968	26	0.11 032	9.89 771	10	12
49	9.78 756	16	9.88 994	26	0.11 006	9.89 761	9	11
50	9.78 772	16	9.89 020	26	0.10 980	9.89 752	10	**10**
51	9.78 788	17	9.89 046	27	0.10 954	9.89 742	10	9
52	9.78 805	16	9.89 073	26	0.10 927	9.89 732	10	8
53	9.78 821	16	9.89 099	26	0.10 901	9.89 722	10	7
54	9.78 837	16	9.89 125	26	0.10 875	9.89 712	10	6
55	9.78 853	16	9.89 151	26	0.10 849	9.89 702	9	5
56	9.78 869	17	9.89 177	26	0.10 823	9.89 693	10	4
57	9.78 886	16	9.89 203	26	0.10 797	9.89 683	10	3
58	9.78 902	16	9.89 229	26	0.10 771	9.89 673	10	2
59	9.78 918	16	9.89 255	26	0.10 745	9.89 663	10	1
60	9.78 934		9.89 281		0.10 719	9.89 653		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	27	26	17	16	10	9
.1	2.7	2.6	1.7	1.6	1.0	0.9
.2	5.4	5.2	3.4	3.2	2.0	1.8
.3	8.1	7.8	5.1	4.8	3.0	2.7
.4	10.8	10.4	6.8	6.4	4.0	3.6
.5	13.5	13.0	8.5	8.0	5.0	4.5
.6	16.2	15.6	10.2	9.6	6.0	5.4
.7	18.9	18.2	11.9	11.2	7.0	6.3
.8	21.6	20.8	13.6	12.8	8.0	7.2
.9	24.3	23.4	15.3	14.4	9.0	8.1

Prop. Pts.

52°

38°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.78 934	16	9.89 281	26	0.10 719	9.89 653	10	60
1	9.78 950	17	9.89 307	26	0.10 693	9.89 643	10	59
2	9.78 967	16	9.89 333	26	0.10 667	9.89 633	10	58
3	9.78 983	16	9.89 359	26	0.10 641	9.89 624	10	57
4	9.78 999	16	9.89 385	26	0.10 615	9.89 614	10	56
5	9.79 015	16	9.89 411	26	0.10 589	9.89 604	10	55
6	9.79 031	16	9.89 437	26	0.10 563	9.89 594	10	54
7	9.79 047	16	9.89 463	26	0.10 537	9.89 584	10	53
8	9.79 063	16	9.89 489	26	0.10 511	9.89 574	10	52
9	9.79 079	16	9.89 515	26	0.10 485	9.89 564	10	51
10	9.79 095	16	9.89 541	26	0.10 459	9.89 554	10	50
11	9.79 111	17	9.89 567	26	0.10 433	9.89 544	10	49
12	9.79 128	16	9.89 593	26	0.10 407	9.89 534	10	48
13	9.79 144	16	9.89 619	26	0.10 381	9.89 524	10	47
14	9.79 160	16	9.89 645	26	0.10 355	9.89 514	10	46
15	9.79 176	16	9.89 671	26	0.10 329	9.89 504	9	45
16	9.79 192	16	9.89 697	26	0.10 303	9.89 495	10	44
17	9.79 208	16	9.89 723	26	0.10 277	9.89 485	10	43
18	9.79 224	16	9.89 749	26	0.10 251	9.89 475	10	42
19	9.79 240	16	9.89 775	26	0.10 225	9.89 465	10	41
20	9.79 256	16	9.89 801	26	0.10 199	9.89 455	10	40
21	9.79 272	16	9.89 827	26	0.10 173	9.89 445	10	39
22	9.79 288	16	9.89 853	26	0.10 147	9.89 435	10	38
23	9.79 304	15	9.89 879	26	0.10 121	9.89 425	10	37
24	9.79 319	16	9.89 905	26	0.10 095	9.89 415	10	36
25	9.79 335	16	9.89 931	26	0.10 069	9.89 405	10	35
26	9.79 351	16	9.89 957	26	0.10 043	9.89 395	10	34
27	9.79 367	16	9.89 983	26	0.10 017	9.89 385	10	33
28	9.79 383	16	9.90 009	26	0.09 991	9.89 375	11	32
29	9.79 399	16	9.90 035	26	0.09 965	9.89 364	10	31
30	9.79 415	16	9.90 061	25	0.09 939	9.89 354	10	30
31	9.79 431	16	9.90 086	26	0.09 914	9.89 344	10	29
32	9.79 447	16	9.90 112	26	0.09 888	9.89 334	10	28
33	9.79 463	15	9.90 138	26	0.09 862	9.89 324	10	27
34	9.79 478	16	9.90 164	26	0.09 836	9.89 314	10	26
35	9.79 494	16	9.90 190	26	0.09 810	9.89 304	10	25
36	9.79 510	16	9.90 216	26	0.09 784	9.89 294	10	24
37	9.79 526	16	9.90 242	26	0.09 758	9.89 284	10	23
38	9.79 542	16	9.90 268	26	0.09 732	9.89 274	10	22
39	9.79 558	15	9.90 294	26	0.09 706	9.89 264	10	21
40	9.79 573	16	9.90 320	26	0.09 680	9.89 254	10	20
41	9.79 589	16	9.90 346	25	0.09 654	9.89 244	11	19
42	9.79 605	16	9.90 371	26	0.09 629	9.89 233	10	18
43	9.79 621	15	9.90 397	26	0.09 603	9.89 223	10	17
44	9.79 636	16	9.90 423	26	0.09 577	9.89 213	10	16
45	9.79 652	16	9.90 449	26	0.09 551	9.89 203	10	15
46	9.79 668	16	9.90 475	26	0.09 525	9.89 193	10	14
47	9.79 684	15	9.90 501	26	0.09 499	9.89 183	10	13
48	9.79 699	16	9.90 527	26	0.09 473	9.89 173	11	12
49	9.79 715	16	9.90 553	25	0.09 447	9.89 162	10	11
50	9.79 731	15	9.90 578	26	0.09 422	9.89 152	10	10
51	9.79 746	16	9.90 604	26	0.09 396	9.89 142	10	9
52	9.79 762	16	9.90 630	26	0.09 370	9.89 132	10	8
53	9.79 778	15	9.90 656	26	0.09 344	9.89 122	10	7
54	9.79 793	16	9.90 682	26	0.09 318	9.89 112	11	6
55	9.79 809	16	9.90 708	26	0.09 292	9.89 101	10	5
56	9.79 825	15	9.90 734	25	0.09 266	9.89 091	10	4
57	9.79 840	16	9.90 759	26	0.09 241	9.89 081	10	3
58	9.79 856	16	9.90 785	26	0.09 215	9.89 071	11	2
59	9.79 872	15	9.90 811	26	0.09 189	9.89 060	10	1
60	9.79 887		9.90 837		0.09 163	9.89 050		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26	25
.1	2.6	2.5
.2	5.2	5.0
.3	7.8	7.5
.4	10.4	10.0
.5	13.0	12.5
.6	15.6	15.0
.7	18.2	17.5
.8	20.8	20.0
.9	23.4	22.5

	17
.1	1.7
.2	3.4
.3	5.1
.4	6.8
.5	8.5
.6	10.2
.7	11.9
.8	13.6
.9	15.3

	16	15
.1	1.6	1.5
.2	3.2	3.0
.3	4.8	4.5
.4	6.4	6.0
.5	8.0	7.5
.6	9.6	9.0
.7	11.2	10.5
.8	12.8	12.0
.9	14.4	13.5

	11
.1	1.1
.2	2.2
.3	3.3
.4	4.4
.5	5.5
.6	6.6
.7	7.7
.8	8.8
.9	9.9

	10	9
.1	1.0	0.9
.2	2.0	1.8
.3	3.0	2.7
.4	4.0	3.6
.5	5.0	4.5
.6	6.0	5.4
.7	7.0	6.3
.8	8.0	7.2
.9	9.0	8.1

Prop. Pts.

51°

39°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.79 887	16	9.90 837	26	0.09 163	9.89 050	10	60
1	9.79 903	15	9.90 863	26	0.09 137	9.89 040	10	59
2	9.79 918	16	9.90 889	25	0.09 111	9.89 030	10	58
3	9.79 934	16	9.90 914	26	0.09 086	9.89 020	11	57
4	9.79 950	15	9.90 940	26	0.09 060	9.89 009	10	56
5	9.79 965	16	9.90 966	26	0.09 034	9.88 999	10	55
6	9.79 981	15	9.90 992	26	0.09 008	9.88 989	11	54
7	9.79 996	16	9.91 018	25	0.08 982	9.88 978	10	53
8	9.80 012	15	9.91 043	26	0.08 957	9.88 968	10	52
9	9.80 027	16	9.91 069	26	0.08 931	9.88 958	10	51
10	9.80 043	15	9.91 095	26	0.08 905	9.88 948	11	50
11	9.80 058	16	9.91 121	26	0.08 879	9.88 937	10	49
12	9.80 074	15	9.91 147	25	0.08 853	9.88 927	10	48
13	9.80 089	16	9.91 172	26	0.08 828	9.88 917	11	47
14	9.80 105	15	9.91 198	26	0.08 802	9.88 906	10	46
15	9.80 120	16	9.91 224	26	0.08 776	9.88 896	10	45
16	9.80 136	15	9.91 250	26	0.08 750	9.88 886	11	44
17	9.80 151	15	9.91 276	25	0.08 724	9.88 875	10	43
18	9.80 166	16	9.91 301	26	0.08 699	9.88 865	10	42
19	9.80 182	15	9.91 327	26	0.08 673	9.88 855	11	41
20	9.80 197	16	9.91 353	26	0.08 647	9.88 844	10	40
21	9.80 213	15	9.91 379	25	0.08 621	9.88 834	10	39
22	9.80 228	16	9.91 404	26	0.08 596	9.88 824	11	38
23	9.80 244	15	9.91 430	26	0.08 570	9.88 813	10	37
24	9.80 259	15	9.91 456	26	0.08 544	9.88 803	10	36
25	9.80 274	16	9.91 482	25	0.08 518	9.88 793	11	35
26	9.80 290	15	9.91 507	26	0.08 493	9.88 782	10	34
27	9.80 305	15	9.91 533	26	0.08 467	9.88 772	11	33
28	9.80 320	16	9.91 559	26	0.08 441	9.88 761	10	32
29	9.80 336	15	9.91 585	25	0.08 415	9.88 751	10	31
30	9.80 351	15	9.91 610	26	0.08 390	9.88 741	11	30
31	9.80 366	16	9.91 636	26	0.08 364	9.88 730	10	29
32	9.80 382	15	9.91 662	26	0.08 338	9.88 720	11	28
33	9.80 397	15	9.91 688	25	0.08 312	9.88 709	10	27
34	9.80 412	16	9.91 713	26	0.08 287	9.88 699	11	26
35	9.80 428	15	9.91 739	26	0.08 261	9.88 688	10	25
36	9.80 443	15	9.91 765	26	0.08 235	9.88 678	10	24
37	9.80 458	15	9 91 791	25	0.08 209	9.88 668	11	23
38	9.80 473	16	9.91 816	26	0.08 184	9.88 657	10	22
39	9.80 489	15	9.91 842	26	0.08 158	9.88 647	11	21
40	9.80 504	15	9.91 868	25	0.08 132	9.88 636	10	20
41	9.80 519	15	9.91 893	26	0.08 107	9.88 626	11	19
42	9.80 534	16	9.91 919	26	0.08 081	9.88 615	10	18
43	9.80 550	15	9.91 945	26	0.08 055	9.88 605	11	17
44	9.80 565	15	9.91 971	25	0.08 029	9.88 594	10	16
45	9.80 580	15	9.91 996	26	0.08 004	9.88 584	11	15
46	9.80 595	15	9.92 022	26	0.07 978	9.88 573	10	14
47	9.80 610	15	9.92 048	25	0.07 952	9.88 563	11	13
48	9.80 625	16	9.92 073	26	0.07 927	9.88 552	10	12
49	9.80 641	15	9.92 099	26	0.07 901	9.88 542	11	11
50	9.80 656	15	9.92 125	25	0.07 875	9.88 531	10	10
51	9.80 671	15	9.92 150	26	0.07 850	9.88 521	11	9
52	9.80 686	15	9.92 176	26	0.07 824	9.88 510	11	8
53	9.80 701	15	9.92 202	25	0.07 798	9.88 499	10	7
54	9.80 716	15	9.92 227	26	0.07 773	9.88 489	11	6
55	9.80 731	15	9.92 253	26	0.07 747	9.88 478	10	5
56	9.80 746	16	9.92 279	25	0.07 721	9.88 468	11	4
57	9.80 762	15	9.92 304	26	0.07 696	9.88 457	10	3
58	9.80 777	15	9.92 330	26	0.07 670	9.88 447	11	2
59	9.80 792	15	9.92 356	25	0.07 644	9.88 436	11	1
60	9.80 807		9.92 381		0.07 619	9.88 425		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

50°

Prop. Pts.

	26	25	16	15	11	10
.1	2.6	2.5	1.6	1.5	1.1	1.0
.2	5.2	5.0	3.2	3.0	2.2	2.0
.3	7.8	7.5	4.8	4.5	3.3	3.0
.4	10.4	10.0	6.4	6.0	4.4	4.0
.5	13.0	12.5	8.0	7.5	5.5	5.0
.6	15.6	15.0	9.6	9.0	6.6	6.0
.7	18.2	17.5	11.2	10.5	7.7	7.0
.8	20.8	20.0	12.8	12.0	8.8	8.0
.9	23.4	22.5	14.4	13.5	9.9	9.0

Prop. Pts.

40°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.80 807	15	9.92 381	26	0.07 619	9.88 425	10	**60**
1	9.80 822	15	9.92 407	26	0.07 593	9.88 415	11	59
2	9.80 837	15	9.92 433	25	0.07 567	9.88 404	10	58
3	9.80 852	15	9.92 458	26	0.07 542	9.88 394	11	57
4	9.80 867	15	9.92 484	26	0.07 516	9.88 383	11	56
5	9.80 882	15	9.92 510	25	0.07 490	9.88 372	10	55
6	9.80 897	15	9.92 535	26	0.07 465	9.88 362	11	54
7	9.80 912	15	9.92 561	26	0.07 439	9.88 351	11	53
8	9.80 927	15	9.92 587	25	0.07 413	9.88 340	10	52
9	9.80 942	15	9.92 612	26	0.07 388	9.88 330	11	51
10	9.80 957	15	9.92 638	25	0.07 362	9.88 319	11	**50**
11	9.80 972	15	9.92 663	26	0.07 337	9.88 308	10	49
12	9.80 987	15	9.92 689	26	0.07 311	9.88 298	11	48
13	9.81 002	15	9.92 715	25	0.07 285	9.88 287	11	47
14	9.81 017	15	9.92 740	26	0.07 260	9.88 276	10	46
15	9.81 032	15	9.92 766	26	0.07 234	9.88 266	11	45
16	9.81 047	14	9.92 792	25	0.07 208	9.88 255	11	44
17	9.81 061	15	9.92 817	26	0.07 183	9.88 244	10	43
18	9.81 076	15	9.92 843	25	0.07 157	9.88 234	11	42
19	9.81 091	15	9.92 868	26	0.07 132	9.88 223	11	41
20	9.81 106	15	9.92 894	26	0.07 106	9.88 212	11	**40**
21	9.81 121	15	9.92 920	25	0.07 080	9.88 201	10	39
22	9.81 136	15	9.92 945	26	0.07 055	9.88 191	11	38
23	9.81 151	15	9.92 971	25	0.07 029	9.88 180	11	37
24	9.81 166	14	9.92 996	26	0.07 004	9.88 169	11	36
25	9.81 180	15	9.93 022	26	0.06 978	9.88 158	10	35
26	9.81 195	15	9.93 048	25	0.06 952	9.88 148	11	34
27	9.81 210	15	9.93 073	26	0.06 927	9.88 137	11	33
28	9.81 225	15	9.93 099	25	0.06 901	9.88 126	11	32
29	9.81 240	14	9.93 124	26	0.06 876	9.88 115	10	31
30	9.81 254	15	9.93 150	25	0.06 850	9.88 105	11	**30**
31	9.81 269	15	9.93 175	26	0.06 825	9.88 094	11	29
32	9.81 284	15	9.93 201	26	0.06 799	9.88 083	11	28
33	9.81 299	15	9.93 227	25	0.06 773	9.88 072	11	27
34	9.81 314	14	9.93 252	26	0.06 748	9.88 061	10	26
35	9.81 328	15	9.93 278	25	0.06 722	9.88 051	11	25
36	9.81 343	15	9.93 303	26	0.06 697	9.88 040	11	24
37	9.81 358	14	9.93 329	25	0.06 671	9.88 029	11	23
38	9.81 372	15	9.93 354	26	0.06 646	9.88 018	11	22
39	9.81 387	15	9.93 380	26	0.06 620	9.88 007	11	21
40	9.81 402	15	9.93 406	25	0.06 594	9.87 996	11	**20**
41	9.81 417	14	9.93 431	26	0.06 569	9.87 985	10	19
42	9.81 431	15	9.93 457	25	0.06 543	9.87 975	11	18
43	9.81 446	15	9.93 482	26	0.06 518	9.87 964	11	17
44	9.81 461	14	9.93 508	25	0.06 492	9.87 953	11	16
45	9.81 475	15	9.93 533	26	0.06 467	9.87 942	11	15
46	9.81 490	15	9.93 559	25	0.06 441	9.87 931	11	14
47	9.81 505	14	9.93 584	26	0.06 416	9.87 920	11	13
48	9.81 519	15	9.93 610	26	0.06 390	9.87 909	11	12
49	9.81 534	15	9.93 636	25	0.06 364	9.87 898	11	11
50	9.81 549	14	9.93 661	26	0.06 339	9.87 887	10	**10**
51	9.81 563	15	9.93 687	25	0.06 313	9.87 877	11	9
52	9.81 578	14	9.93 712	26	0.06 288	9.87 866	11	8
53	9.81 592	15	9.93 738	25	0.06 262	9.87 855	11	7
54	9.81 607	15	9.93 763	26	0.06 237	9.87 844	11	6
55	9.81 622	14	9.93 789	25	0.06 211	9.87 833	11	5
56	9.81 636	15	9.93 814	26	0.06 186	9.87 822	11	4
57	9.81 651	14	9.93 840	25	0.06 160	9.87 811	11	3
58	9.81 665	15	9.93 865	26	0.06 135	9.87 800	11	2
59	9.81 680	14	9.93 891	25	0.06 109	9.87 789	11	1
60	9.81 694		9.93 916		0.06 084	9.87 778		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26
.1	2.6
.2	5.2
.3	7.8
.4	10.4
.5	13.0
.6	15.6
.7	18.2
.8	20.8
.9	23.4

	25
.1	2.5
.2	5.0
.3	7.5
.4	10.0
.5	12.5
.6	15.0
.7	17.5
.8	20.0
.9	22.5

	15
.1	1.5
.2	3.0
.3	4.5
.4	6.0
.5	7.5
.6	9.0
.7	10.5
.8	12.0
.9	13.5

	14
.1	1.4
.2	2.8
.3	4.2
.4	5.6
.5	7.0
.6	8.4
.7	9.8
.8	11.2
.9	12.6

	11	10
.1	1.1	1.0
.2	2.2	2.0
.3	3.3	3.0
.4	4.4	4.0
.5	5.5	5.0
.6	6.6	6.0
.7	7.7	7.0
.8	8.8	8.0
.9	9.9	9.0

Prop. Pts.

49°

41°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.81 694	15	9.93 916	26	0.06 084	9.87 778	11	60
1	9.81 709	14	9.93 942	25	0.06 058	9.87 767	11	59
2	9.81 723	15	9.93 967	26	0.06 033	9.87 756	11	58
3	9.81 738	14	9.93 993	25	0.06 007	9.87 745	11	57
4	9.81 752	15	9.94 018	26	0.05 982	9.87 734	11	56
5	9.81 767	14	9.94 044	25	0.05 956	9.87 723	11	55
6	9.81 781	15	9.94 069	26	0.05 931	9.87 712	11	54
7	9.81 796	14	9.94 095	25	0.05 905	9.87 701	11	53
8	9.81 810	15	9.94 120	26	0.05 880	9.87 690	11	52
9	9.81 825	14	9.94 146	25	0.05 854	9.87 679	11	51
10	9.81 839	15	9.94 171	26	0.05 829	9.87 668	11	50
11	9.81 854	14	9.94 197	25	0.05 803	9.87 657	11	49
12	9.81 868	14	9.94 222	26	0.05 778	9.87 646	11	48
13	9.81 882	15	9.94 248	25	0.05 752	9.87 635	11	47
14	9.81 897	14	9.94 273	26	0.05 727	9.87 624	11	46
15	9.81 911	15	9.94 299	25	0.05 701	9.87 613	12	45
16	9.81 926	14	9.94 324	26	0.05 676	9.87 601	11	44
17	9.81 940	15	9.94 350	25	0.05 650	9.87 590	11	43
18	9.81 955	14	9.94 375	26	0.05 625	9.87 579	11	42
19	9.81 969	14	9.94 401	25	0.05 599	9.87 568	11	41
20	9.81 983	15	9.94 426	26	0.05 574	9.87 557	11	40
21	9.81 998	14	9.94 452	25	0.05 548	9.87 546	11	39
22	9.82 012	14	9.94 477	26	0.05 523	9.87 535	11	38
23	9.82 026	15	9.94 503	25	0.05 497	9.87 524	11	37
24	9.82 041	14	9.94 528	26	0.05 472	9.87 513	12	36
25	9.82 055	14	9.94 554	25	0.05 446	9.87 501	11	35
26	9.82 069	15	9.94 579	25	0.05 421	9.87 490	11	34
27	9.82 084	14	9.94 604	26	0.05 396	9.87 479	11	33
28	9.82 098	14	9.94 630	25	0.05 370	9.87 468	11	32
29	9.82 112	14	9.94 655	26	0.05 345	9.87 457	11	31
30	9.82 126	15	9.94 681	25	0.05 319	9.87 446	12	30
31	9.82 141	14	9.94 706	26	0.05 294	9.87 434	11	29
32	9.82 155	14	9.94 732	25	0.05 268	9.87 423	11	28
33	9.82 169	15	9.94 757	26	0.05 243	9.87 412	11	27
34	9.82 184	14	9.94 783	25	0.05 217	9.87 401	11	26
35	9.82 198	14	9.94 808	26	0.05 192	9.87 390	12	25
36	9.82 212	14	9.94 834	25	0.05 166	9.87 378	11	24
37	9.82 226	14	9.94 859	25	0.05 141	9.87 367	11	23
38	9.82 240	15	9.94 884	26	0.05 116	9.87 356	11	22
39	9.82 255	14	9.94 910	25	0.05 090	9.87 345	11	21
40	9.82 269	14	9.94 935	26	0.05 065	9.87 334	12	20
41	9.82 283	14	9.94 961	25	0.05 039	9.87 322	11	19
42	9.82 297	14	9.94 986	26	0.05 014	9.87 311	11	18
43	9.82 311	15	9.95 012	25	0.04 988	9.87 300	12	17
44	9.82 326	14	9.95 037	25	0.04 963	9.87 288	11	16
45	9.82 340	14	9.95 062	26	0.04 938	9.87 277	11	15
46	9.82 354	14	9.95 088	25	0.04 912	9.87 266	11	14
47	9.82 368	14	9.95 113	26	0.04 887	9.87 255	12	13
48	9.82 382	14	9.95 139	25	0.04 861	9.87 243	11	12
49	9.82 396	14	9.95 164	26	0.04 836	9.87 232	11	11
50	9.82 410	14	9.95 190	25	0.04 810	9.87 221	12	10
51	9.82 424	15	9.95 215	25	0.04 785	9.87 209	11	9
52	9.82 439	14	9.95 240	26	0.04 760	9.87 198	11	8
53	9.82 453	14	9.95 266	25	0.04 734	9.87 187	12	7
54	9.82 467	14	9.95 291	26	0.04 709	9.87 175	11	6
55	9.82 481	14	9.95 317	25	0.04 683	9.87 164	11	5
56	9.82 495	14	9.95 342	26	0.04 658	9.87 153	12	4
57	9.82 509	14	9.95 368	25	0.04 632	9.87 141	11	3
58	9.82 523	14	9.95 393	25	0.04 607	9.87 130	11	2
59	9.82 537	14	9.95 418	26	0.04 582	9.87 119	12	1
60	9.82 551		9.95 444		0.04 556	9.87 107		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

48°

Prop. Pts.

	26
.1	2.6
.2	5.2
.3	7.8
.4	10.4
.5	13.0
.6	15.6
.7	18.2
.8	20.8
.9	23.4

	25
.1	2.5
.2	5.0
.3	7.5
.4	10.0
.5	12.5
.6	15.0
.7	17.5
.8	20.0
.9	22.5

	15
1	1.5
.2	3.0
.3	4.5
.4	6.0
.5	7.5
.6	9.0
.7	10.5
.8	12.0
.9	13.5

	14
.1	1.4
.2	2.8
.3	4.2
.4	5.6
.5	7.0
.6	8.4
.7	9.8
.8	11.2
.9	12.6

	12	11
.1	1.2	1.1
.2	2.4	2.2
.3	3.6	3.3
.4	4.8	4.4
.5	6.0	5.5
.6	7.2	6.6
.7	8.4	7.7
.8	9.6	8.8
.9	10.8	9.9

Prop. Pts.

42°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.82 551	14	9.95 444	25	0.04 556	9.87 107	11	**60**
1	9.82 565	14	9.95 469	26	0.04 531	9.87 096	11	59
2	9.82 579	14	9.95 495̄	25	0.04 505	9.87 085̄	12	58
3	9.82 593	14	9.95 520	25	0.04 480	9.87 073	11	57
4	9.82 607	14	9.95 545	26	0.04 455̄	9.87 062	12	56
5	9.82 621	14	9.95 571	25	0.04 429	9.87 050	11	55
6	9.82 635	14	9.95 596	26	0.04 404	9.87 039	11	54
7	9.82 649	14	9.95 622	25	0.04 378	9.87 028	12	53
8	9.82 663	14	9.95 647	25	0.04 353	9.87 016	11	52
9	9.82 677	14	9.95 672	26	0.04 328	9.87 005̄	12	51
10	9.82 691	14	9.95 698	25	0.04 302	9.86 993	11	**50**
11	9.82 705̄	14	9.95 723	25	0.04 277	9.86 982	12	49
12	9.82 719	14	9.95 748	26	0.04 252	9.86 970	11	48
13	9.82 733	14	9.95 774	25	0.04 226	9.86 959	12	47
14	9.82 747	14	9.95 799	26	0.04 201	9 86 947	11	46
15	9.82 761	14	9.95 825̄	25	0.04 175	9.86 936	12	45
16	9.82 775̄	13	9.95 850	25	0.04 15̄0	9.86 924	11	44
17	9.82 788	14	9.95 875	26	0.04 125̄	9.86 913	11	43
18	9.82 802	14	9.95 901	25	0.04 099	9.86 902	12	42
19	9.82 816	14	9.95 926	26	0.04 074	9.86 890	11	41
20	9.82 830	14	9.95 952	25	0.04 048	9.86 879	12	**40**
21	9.82 844	14	9.95 977	25	0.04 023	9.86 867	12	39
22	9.82 858	14	9.96 002	26	0.03 998	9.86 855	11	38
23	9.82 872	13	9.96 028	25	0.03 972	9.86 844	12	37
24	9.82 885	14	9.96 053	25	0.03 947	9.86 832	11	36
25	9.82 899	14	9.96 078	26	0.03 922	9.86 821	12	35
26	9.82 913	14	9.96 104	25	0.03 896	9.86 809	11	34
27	9.82 927	14	9.96 129	26	0.03 871	9.86 798	12	33
28	9.82 941	14	9.96 155̄	25	0.03 845	9.86 786	11	32
29	9.82 955̄	13	9:96 180	25	0.03 820	9.86 775̄	12	31
30	9.82 968	14	9.96 205	26	0.03 795̄	9.86 763	11	**30**
31	9.82 982	14	9.96 231	25	0.03 769	9.86 752	12	29
32	9.82 996	14	9.96 256	25	0.03 744	9.86 740	12	28
33	9.83 010	13	9.96 281	26	0.03 719	9.86 728	11	27
34	9.83 023	14	9.96 307	25	0.03 693	9.86 717	12	26
35	9.83 037	14	9.96 332	25	0.03 668	9.86 705	11	25
36	9.83 051	14	9.96 357	26	0.03 643	9.86 694	12	24
37	9.83 065̄	13	9.96 383	25	0.03 617	9.86 682	12	23
38	9.83 078	14	9.96 408	25	0.03 592	9.86 670	11	22
39	9.83 092	14	9.96 433	26	0.03 567	9.86 659	12	21
40	9.83 106	14	9.96 459	25	0.03 541	9.86 647	12	**20**
41	9.83 120	13	9.96 484	26	0.03 516	9.86 635	11	19
42	9.83 133	14	9.96 510	25	0.03 490	9.86 624	12	18
43	9.83 147	14	9.96 535̄	25	0.03 465	9.86 612	12	17
44	9.83 161	13	9.96 560	26	0.03 440	9.86 600	11	16
45	9.83 174	14	9.96 586	25	0.03 414	9.86 589	12	15
46	9.83 188	14	9.96 611	25	0.03 389	9.86 577	12	14
47	9.83 202	13	9.96 636	26	0.03 364	9.86 565	11	13
48	9.83 215	14	9.96 662	25	0.03 338	9.86 554	12	12
49	9.83 229	13	9.96 687	25	0.03 313	9.86 542	12	11
50	9.83 242	14	9.96 712	26	0.03 288	9.86 530	12	**10**
51	9.83 256	14	9.96 738	25	0.03 262	9.86 518	11	9
52	9.83 270	13	9.96 763	25	0.03 237	9.86 507	12	8
53	9.83 283	14	9.96 788	26	0.03 212	9.86 495	12	7
54	9.83 297	13	9.96 814	25	0.03 186	9.86 483	11	6
55	9.83 310	14	9.96 839	25	0.03 161	9.86 472	12	5
56	9.83 324	14	9.96 864	26	0.03 136	9.86 460	12	4
57	9.83 338	13	9.96 890	25	0.03 110	9.86 448	12	3
58	9.83 351	14	9.96 915̄	25	0.03 085	9.86 436	11	2
59	9.83 365̄	13	9.96 940	26	0.03 060	9.86 425̄	12	1
60	9.83 378		9.96 966		0.03 034	9.86 413		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26
.1	2.6
.2	5.2
.3	7.8
.4	10.4
.5	13.0
.6	15.6
.7	18.2
.8	20.8
.9	23.4

	25
.1	2.5
.2	5.0
.3	7.5
.4	10.0
.5	12.5
.6	15.0
.7	17.5
.8	20.0
.9	22.5

	14
.1	1.4
.2	2.8
.3	4.2
.4	5.6
.5	7.0
.6	8.4
.7	9.8
.8	11.2
.9	12.6

	13
.1	1.3
.2	2.6
.3	3.9
.4	5.2
.5	6.5
.6	7.8
.7	9.1
.8	10.4
.9	11.7

	12	11
.1	1.2	1.1
.2	2.4	2.2
3	3.6	3.3
.4	4.8	4.4
.5	6.0	5.5
.6	7.2	6.6
.7	8.4	7.7
.8	9.6	8.8
.9	10.8	9.9

Prop. Pts.

47°

43°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.83 378	14	9.96 966	25	0.03 034	9.86 413	12	60
1	9.83 392	13	9.96 991	25	0.03 009	9.86 401	12	59
2	9.83 405	14	9.97 016	26	0.02 984	9.86 389	12	58
3	9.83 419	13	9.97 042	25	0.02 958	9.86 377	11	57
4	9.83 432	14	9.97 067	25	0.02 933	9.86 366	12	56
5	9.83 446	13	9.97 092	26	0.02 908	9.86 354	12	55
6	9.83 459	14	9.97 118	25	0.02 882	9.86 342	12	54
7	9.83 473	13	9.97 143	25	0.02 857	9.86 330	12	53
8	9.83 486	14	9.97 168	25	0.02 832	9.86 318	12	52
9	9.83 500	13	9.97 193	26	0.02 807	9.86 306	11	51
10	9.83 513	14	9.97 219	25	0.02 781	9.86 295	12	50
11	9.83 527	13	9.97 244	25	0.02 756	9.86 283	12	49
12	9.83 540	14	9.97 269	26	0.02 731	9.86 271	12	48
13	9.83 554	13	9.97 295	25	0.02 705	9.86 259	12	47
14	9.83 567	14	9.97 320	25	0.02 680	9.86 247	12	46
15	9.83 581	13	9.97 345	26	0.02 655	9.86 235	12	45
16	9.83 594	14	9.97 371	25	0.02 629	9.86 223	12	44
17	9.83 608	13	9.97 396	25	0.02 604	9.86 211	11	43
18	9.83 621	13	9.97 421	26	0.02 579	9.86 200	12	42
19	9.83 634	14	9.97 447	25	0.02 553	9.86 188	12	41
20	9.83 648	13	9.97 472	25	0.02 528	9.86 176	12	40
21	9.83 661	13	9.97 497	26	0.02 503	9.86 164	12	39
22	9.83 674	14	9.97 523	25	0.02 477	9.86 152	12	38
23	9.83 688	13	9.97 548	25	0.02 452	9.86 140	12	37
24	9.83 701	14	9.97 573	25	0.02 427	9.86 128	12	36
25	9.83 715	13	9.97 598	26	0.02 402	9.86 116	12	35
26	9.83 728	13	9.97 624	25	0.02 376	9.86 104	12	34
27	9.83 741	14	9.97 649	25	0.02 351	9.86 092	12	33
28	9.83 755	13	9.97 674	26	0.02 326	9.86 080	12	32
29	9.83 768	13	9.97 700	25	0.02 300	9.86 068	12	31
30	9.83 781	14	9.97 725	25	0.02 275	9.86 056	12	30
31	9.83 795	13	9.97 750	26	0.02 250	9.86 044	12	29
32	9.83 808	13	9.97 776	25	0.02 224	9.86 032	12	28
33	9.83 821	13	9.97 801	25	0.02 199	9.86 020	12	27
34	9.83 834	14	9.97 826	25	0.02 174	9.86 008	12	26
35	9.83 848	13	9.97 851	26	0.02 149	9.85 996	12	25
36	9.83 861	13	9.97 877	25	0.02 123	9.85 984	12	24
37	9.83 874	13	9.97 902	25	0.02 098	9.85 972	12	23
38	9.83 887	14	9.97 927	26	0.02 073	9.85 960	12	22
39	9.83 901	13	9.97 953	25	0.02 047	9.85 948	12	21
40	9.83 914	13	9.97 978	25	0.02 022	9.85 936	12	20
41	9.83 927	13	9.98 003	26	0.01 997	9.85 924	12	19
42	9.83 940	14	9.98 029	25	0.01 971	9.85 912	12	18
43	9.83 954	13	9.98 054	25	0.01 946	9.85 900	12	17
44	9.83 967	13	9.98 079	25	0.01 921	9.85 888	12	16
45	9.83 980	13	9.98 104	26	0.01 896	9.85 876	12	15
46	9.83 993	13	9.98 130	25	0.01 870	9.85 864	13	14
47	9.84 006	14	9.98 155	25	0.01 845	9.85 851	12	13
48	9.84 020	13	9.98 180	26	0.01 820	9.85 839	12	12
49	9.84 033	13	9.98 206	25	0.01 794	9.85 827	12	11
50	9.84 046	13	9.98 231	25	0.01 769	9.85 815	12	10
51	9.84 059	13	9.98 256	25	0.01 744	9.85 803	12	9
52	9.84 072	13	9.98 281	26	0.01 719	9.85 791	12	8
53	9.84 085	13	9.98 307	25	0.01 693	9.85 779	13	7
54	9.84 098	14	9.98 332	25	0.01 668	9.85 766	12	6
55	9.84 112	13	9.98 357	26	0.01 643	9.85 754	12	5
56	9.84 125	13	9.98 383	25	0.01 617	9.85 742	12	4
57	9.84 138	13	9.98 408	25	0.01 592	9.85 730	12	3
58	9.84 151	13	9.98 433	25	0.01 567	9.85 718	12	2
59	9.84 164	13	9.98 458	26	0.01 542	9.85 706	13	1
60	9.84 177		9.98 484		0.01 516	9.85 693		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26
.1	2.6
.2	5.2
.3	7.8
.4	10.4
.5	13.0
.6	15.6
.7	18.2
.8	20.8
.9	23.4

	25
.1	2.5
.2	5.0
.3	7.5
.4	10.0
.5	12.5
.6	15.0
.7	17.5
.8	20.0
.9	22.5

	14
.1	1.4
.2	2.8
.3	4.2
.4	5.6
.5	7.0
.6	8.4
.7	9.8
.8	11.2
.9	12.6

	13
.1	1.3
.2	2.6
.3	3.9
.4	5.2
.5	6.5
.6	7.8
.7	9.1
.8	10.4
.9	11.7

	12	11
.1	1.2	1.1
.2	2.4	2.2
.3	3.6	3.3
.4	4.8	4.4
.5	6.0	5.5
.6	7.2	6.6
.7	8.4	7.7
.8	9.6	8.8
.9	10.8	9.9

Prop. Pts.

46°

44°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.84 177	13	9.98 484	25	0.01 516	9.85 693	12	60
1	9.84 190	13	9.98 509	25	0.01 491	9.85 681	12	59
2	9.84 203	13	9.98 534	26	0.01 466	9.85 669	12	58
3	9.84 216	13	9.98 560	25	0.01 440	9.85 657	12	57
4	9.84 229	13	9.98 585̄	25	0.01 415	9.85 645̄	13	56
5	9.84 242	13	9.98 610	25	0.01 390	9.85 632	12	55
6	9.84 255	14	9.98 635	26	0.01 365̄	9.85 620	12	54
7	9.84 269	13	9.98 661	25	0.01 339	9.85 608	12	53
8	9.84 282	13	9.98 686	25	0.01 314	9.85 596	13	52
9	9.84 295̄	13	9.98 711	26	0.01 289	9.85 583	12	51
10	9.84 308	13	9.98 737	25	0.01 263	9.85 571	12	50
11	9.84 321	13	9.98 762	25	0.01 238	9.85 559	12	49
12	9.84 334	13	9.98 787	25	0.01 213	9.85 547	13	48
13	9.84 347	13	9.98 812	26	0.01 188	9.85 534	12	47
14	9.84 360	13	9.98 838	25	0.01 162	9.85 522	12	46
15	9.84 373	12	9.98 863	25	0.01 137	9.85 510	13	45
16	9.84 385	13	9.98 888	25	0.01 112	9.85 497	12	44
17	9.84 398	13	9.98 913	26	0.01 087	9.85 485̄	12	43
18	9.84 411	13	9.98 939	25	0.01 061	9.85 473	13	42
19	9.84 424	13	9.98 964	25	0.01 036	9.85 460	12	41
20	9.84 437	13	9.98 989	26	0.01 011	9.85 448	12	40
21	9.84 450	13	9.99 015̄	25	0.00 985	9.85 436	13	39
22	9.84 463	13	9.99 040	25	0.00 960	9.85 423	12	38
23	9.84 476	13	9.99 065	25	0.00 935̄	9.85 411	12	37
24	9.84 489	13	9.99 090	26	0.00 910	9.85 399	13	36
25	9.84 502	13	9.99 116	25	0.00 884	9.85 386	12	35
26	9.84 515̄	13	9.99 141	25	0.00 859	9.85 374	13	34
27	9.84 528	12	9.99 166	25	0.00 834	9.85 361	12	33
28	9.84 540	13	9.99 191	26	0.00 809	9.85 349	12	32
29	9.84 553	13	9.99 217	25	0.00 783	9.85 337	13	31
30	9.84 566	13	9.99 242	25	0.00 758	9.85 324	12	30
31	9.84 579	13	9.99 267	26	0.00 733	9.85 312	13	29
32	9.84 592	13	9.99 293	25	0.00 707	9.85 299	12	28
33	9.84 605̄	13	9.99 318	25	0.00 682	9.85 287	13	27
34	9.84 618	12	9.99 343	25	0.00 657	9.85 274	12	26
35	9.84 630	13	9.99 368	26	0.00 632	9.85 262	12	25
36	9.84 643	13	9.99 394	25	0.00 606	9.85 25̄0	13	24
37	9.84 656	13	9.99 419	25	0.00 581	9.85 237	12	23
38	9.84 669	13	9.99 444	25	0.00 556	9.85 225̄	13	22
39	9.84 682	12	9.99 469	26	0.00 531	9.85 212	12	21
40	9.84 694	13	9.99 495̄	25	0.00 505	9.85 200	13	20
41	9.84 707	13	9.99 520	25	0.00 480	9.85 187	12	19
42	9.84 720	13	9.99 545	25	0.00 455̄	9.85 175̄	13	18
43	9.84 733	12	9.99 570	26	0.00 430	9.85 162	12	17
44	9.84 745	13	9.99 596	25	0.00 404	9.85 15̄0	13	16
45	[illegible]84 758	13	9.99 621	25	0.00 379	9.85 137	12	15
46	9.84 771	13	9.99 646	26	0.00 354	9.85 125̄	13	14
47	9.84 784	12	9.99 672	25	0.00 328	9.85 112	12	13
48	9.84 796	13	9.99 697	25	0.00 303	9.85 100	13	12
49	9.84 809	13	9.99 722	25	0.00 278	9.85 087	13	11
50	9.84 822	13	9.99 747	26	0.00 253	9.85 074	12	10
51	9.84 835̄	12	9.99 773	25	0.00 227	9.85 062	13	9
52	9.84 847	13	9.99 798	25	0.00 202	9.85 049	12	8
53	9.84 860	13	9.99 823	25	0.00 177	9.85 037	13	7
54	9.84 873	12	9.99 848	26	0.00 152	9.85 024	12	6
55	9.84 885	13	9.99 874	25	0.00 126	9.85 012	13	5
56	9.84 898	13	9.99 899	25	0.00 101	9.84 999	13	4
57	9.84 911	12	9.99 924	25	0.00 076	9.84 986	12	3
58	9.84 923	13	9.99 949	26	0.00 051	9.84 974	13	2
59	9.84 936	13	9.99 975̄	25	0.00 025	9.84 961	12	1
60	9.84 949		0.00 000		0.00 000	9.84 949		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26	25	14	13	12
.1	2.6	2.5	1.4	1.3	1.2
.2	5.2	5.0	2.8	2.6	2.4
.3	7.8	7.5	4.2	3.9	3.6
.4	10.4	10.0	5.6	5.2	4.8
.5	13.0	12.5	7.0	6.5	6.0
.6	15.6	15.0	8.4	7.8	7.2
.7	18.2	17.5	9.8	9.1	8.4
.8	20.8	20.0	11.2	10.4	9.6
.9	23.4	22.5	12.6	11.7	10.8

Prop. Pts.

45°

TABLE V.

NATURAL

SINES AND COSINES.

	0°		1°		2°		3°		4°		
′	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	.00000	1.00000	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.00029	1.00000	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.00058	1.00000	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.00087	1.00000	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.00116	1.00000	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.00145	1.00000	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.00175	1.00000	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.00204	1.00000	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.00233	1.00000	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.00262	1.00000	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.00291	1.00000	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.00320	.99999	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.00349	.99999	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.00378	.99999	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.00407	.99999	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.00465	.99999	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.00495	.99999	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.00524	.99999	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.00553	.99998	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.00611	.99998	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.00640	.99998	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.00669	.99998	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.00698	.99998	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.00756	.99997	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.00785	.99997	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.00814	.99997	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.00844	.99996	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.00902	.99996	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.00931	.99996	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.00960	.99995	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.00989	.99995	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.01047	.99995	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.01076	.99994	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.01105	.99994	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.01134	.99994	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.01193	.99993	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.01222	.99993	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.01251	.99992	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.01280	.99992	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.01338	.99991	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.01367	.99991	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.01396	.99990	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.01425	.99990	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.01483	.99989	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.01513	.99989	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.01542	.99988	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.01571	.99988	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.01629	.99987	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.01658	.99986	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.01687	.99986	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.01716	.99985	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	′
	89°		88°		87°		86°		85°		

	5°		6°		7°		8°		9°		
′	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	.15672	.98764	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	.15701	.98760	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	.15730	.98755	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	.15758	.98751	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	.15787	.98746	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	.15816	.98741	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	.15845	.98737	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	.15873	.98732	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	.15902	.98728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	.15959	.98718	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	.15988	.98714	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	.16017	.98709	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	.16046	.98704	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	.16074	.98700	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	.16103	.98695	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	.16132	.98690	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	.16160	.98686	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	.16189	.98681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	.16246	.98671	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	.16275	.98667	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	.16304	.98662	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	.16333	.98657	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	.16361	.98652	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	.16390	.98648	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	.16419	.98643	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	.16447	.98638	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	.16476	.98633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	.16533	.98624	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	.16562	.98619	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	.16591	.98614	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	.16620	.98609	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	.16648	.98604	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	.16677	.98600	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	.16706	.98595	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	.16734	.98590	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	.16763	.98585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	.16820	.98575	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	.16849	.98570	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	.16878	.98565	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	.16906	.98561	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	.16935	.98556	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	.16964	.98551	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	.16992	.98546	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	.17021	.98541	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	.17050	.98536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	.17107	.98526	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	.17136	.98521	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	.17164	.98516	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	.17193	.98511	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	.17222	.98506	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	.17250	.98501	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	.17279	.98496	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	.17308	.98491	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	.17336	.98486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	0
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	′
	84°		83°		82°		81°		80°		

	10°		11°		12°		13°		14°		
′	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	60
1	.17393	.98476	.19109	.98157	.20820	.97809	.22523	.97430	.24220	.97023	59
2	.17422	.98471	.19138	.98152	.20848	.97803	.22552	.97424	.24249	.97015	58
3	.17451	.98466	.19167	.98146	.20877	.97797	.22580	.97417	.24277	.97008	57
4	.17479	.98461	.19195	.98140	.20905	.97791	.22608	.97411	.24305	.97001	56
5	.17508	.98455	.19224	.98135	.20933	.97784	.22637	.97404	.24333	.96994	55
6	.17537	.98450	.19252	.98129	.20962	.97778	.22665	.97398	.24362	.96987	54
7	.17565	.98445	.19281	.98124	.20990	.97772	.22693	.97391	.24390	.96980	53
8	.17594	.98440	.19309	.98118	.21019	.97766	.22722	.97384	.24418	.96973	52
9	.17623	.98435	.19338	.98112	.21047	.97760	.22750	.97378	.24446	.96966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	50
11	.17680	.98425	.19395	.98101	.21104	.97748	.22807	.97365	.24503	.96952	49
12	.17708	.98420	.19423	.98096	.21132	.97742	.22835	.97358	.24531	.96945	48
13	.17737	.98414	.19452	.98090	.21161	.97735	.22863	.97351	.24559	.96937	47
14	.17766	.98409	.19481	.98084	.21189	.97729	.22892	.97345	.24587	.96930	46
15	.17794	.98404	.19509	.98079	.21218	.97723	.22920	.97338	.24615	.96923	45
16	.17823	.98399	.19538	.98073	.21246	.97717	.22948	.97331	.24644	.96916	44
17	.17852	.98394	.19566	.98067	.21275	.97711	.22977	.97325	.24672	.96909	43
18	.17880	.98389	.19595	.98061	.21303	.97705	.23005	.97318	.24700	.96902	42
19	.17909	.98383	.19623	.98056	.21331	.97698	.23033	.97311	.24728	.96894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	40
21	.17966	.98373	.19680	.98044	.21388	.97686	.23090	.97298	.24784	.96880	39
22	.17995	.98368	.19709	.98039	.21417	.97680	.23118	.97291	.24813	.96873	38
23	.18023	.98362	.19737	.98033	.21445	.97673	.23146	.97284	.24841	.96866	37
24	.18052	.98357	.19766	.98027	.21474	.97667	.23175	.97278	.24869	.96858	36
25	.18081	.98352	.19794	.98021	.21502	.97661	.23203	.97271	.24897	.96851	35
26	.18109	.98347	.19823	.98016	.21530	.97655	.23231	.97264	.24925	.96844	34
27	.18138	.98341	.19851	.98010	.21559	.97648	.23260	.97257	.24954	.96837	33
28	.18166	.98336	.19880	.98004	.21587	.97642	.23288	.97251	.24982	.96829	32
29	.18195	.98331	.19908	.97998	.21616	.97636	.23316	.97244	.25010	.96822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	30
31	.18252	.98320	.19965	.97987	.21672	.97623	.23373	.97230	.25066	.96807	29
32	.18281	.98315	.19994	.97981	.21701	.97617	.23401	.97223	.25094	.96800	28
33	.18309	.98310	.20022	.97975	.21729	.97611	.23429	.97217	.25122	.96793	27
34	.18338	.98304	.20051	.97969	.21758	.97604	.23458	.97210	.25151	.96786	26
35	.18367	.98299	.20079	.97963	.21786	.97598	.23486	.97203	.25179	.96778	25
36	.18395	.98294	.20108	.97958	.21814	.97592	.23514	.97196	.25207	.96771	24
37	.18424	.98288	.20136	.97952	.21843	.97585	.23542	.97189	.25235	.96764	23
38	.18452	.98283	.20165	.97946	.21871	.97579	.23571	.97182	.25263	.96756	22
39	.18481	.98277	.20193	.97940	.21899	.97573	.23599	.97176	.25291	.96749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	20
41	.18538	.98267	.20250	.97928	.21956	.97560	.23656	.97162	.25348	.96734	19
42	.18567	.98261	.20279	.97922	.21985	.97553	.23684	.97155	.25376	.96727	18
43	.18595	.98256	.20307	.97916	.22013	.97547	.23712	.97148	.25404	.96719	17
44	.18624	.98250	.20336	.97910	.22041	.97541	.23740	.97141	.25432	.96712	16
45	.18652	.98245	.20364	.97905	.22070	.97534	.23769	.97134	.25460	.96705	15
46	.18681	.98240	.20393	.97899	.22098	.97528	.23797	.97127	.25488	.96697	14
47	.18710	.98234	.20421	.97893	.22126	.97521	.23825	.97120	.25516	.96690	13
48	.18738	.98229	.20450	.97887	.22155	.97515	.23853	.97113	.25545	.96682	12
49	.18767	.98223	.20478	.97881	.22183	.97508	.23882	.97106	.25573	.96675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	10
51	.18824	.98212	.20535	.97869	.22240	.97496	.23938	.97093	.25629	.96660	9
52	.18852	.98207	.20563	.97863	.22268	.97489	.23966	.97086	.25657	.96653	8
53	.18881	.98201	.20592	.97857	.22297	.97483	.23995	.97079	.25685	.96645	7
54	.18910	.98196	.20620	.97851	.22325	.97476	.24023	.97072	.25713	.96638	6
55	.18938	.98190	.20649	.97845	.22353	.97470	.24051	.97065	.25741	.96630	5
56	.18967	.98185	.20677	.97839	.22382	.97463	.24079	.97058	.25769	.96623	4
57	.18995	.98179	.20706	.97833	.22410	.97457	.24108	.97051	.25798	.96615	3
58	.19024	.98174	.20734	.97827	.22438	.97450	.24136	.97044	.25826	.96608	2
59	.19052	.98168	.20763	.97821	.22467	.97444	.24164	.97037	.25854	.96600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	0
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	′
	79°		78°		77°		76°		75°		

	15°		16°		17°		18°		19°		
′	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	.25882	.96593	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	60
1	.25910	.96585	.27592	.96118	.29265	.95622	.30929	.95097	.32584	.94542	59
2	.25938	.96578	.27620	.96110	.29293	.95613	.30957	.95088	.32612	.94533	58
3	.25966	.96570	.27648	.96102	.29321	.95605	.30985	.95079	.32639	.94523	57
4	.25994	.96562	.27676	.96094	.29348	.95596	.31012	.95070	.32667	.94514	56
5	.26022	.96555	.27704	.96086	.29376	.95588	.31040	.95061	.32694	.94504	55
6	.26050	.96547	.27731	.96078	.29404	.95579	.31068	.95052	.32722	.94495	54
7	.26079	.96540	.27759	.96070	.29432	.95571	.31095	.95043	.32749	.94485	53
8	.26107	.96532	.27787	.96062	.29460	.95562	.31123	.95033	.32777	.94476	52
9	.26135	.96524	.27815	.96054	.29487	.95554	.31151	.95024	.32804	.94466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	50
11	.26191	.96509	.27871	.96037	.29543	.95536	.31206	.95006	.32859	.94447	49
12	.26219	.96502	.27899	.96029	.29571	.95528	.31233	.94997	.32887	.94438	48
13	.26247	.96494	.27927	.96021	.29599	.95519	.31261	.94988	.32914	.94428	47
14	.26275	.96486	.27955	.96013	.29626	.95511	.31289	.94979	.32942	.94418	46
15	.26303	.96479	.27983	.96005	.29654	.95502	.31316	.94970	.32969	.94409	45
16	.26331	.96471	.28011	.95997	.29682	.95493	.31344	.94961	.32997	.94399	44
17	.26359	.96463	.28039	.95989	.29710	.95485	.31372	.94952	.33024	.94390	43
18	.26387	.96456	.28067	.95981	.29737	.95476	.31399	.94943	.33051	.94380	42
19	.26415	.96448	.28095	.95972	.29765	.95467	.31427	.94933	.33079	.94370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	40
21	.26471	.96433	.28150	.95956	.29821	.95450	.31482	.94915	.33134	.94351	39
22	.26500	.96425	.28178	.95948	.29849	.95441	.31510	.94906	.33161	.94342	38
23	.26528	.96417	.28206	.95940	.29876	.95433	.31537	.94897	.33189	.94332	37
24	.26556	.96410	.28234	.95931	.29904	.95424	.31565	.94888	.33216	.94322	36
25	.26584	.96402	.28262	.95923	.29932	.95415	.31593	.94878	.33244	.94313	35
26	.26612	.96394	.28290	.95915	.29960	.95407	.31620	.94869	.33271	.94303	34
27	.26640	.96386	.28318	.95907	.29987	.95398	.31648	.94860	.33298	.94293	33
28	.26668	.96379	.28346	.95898	.30015	.95389	.31675	.94851	.33326	.94284	32
29	.26696	.96371	.28374	.95890	.30043	.95380	.31703	.94842	.33353	.94274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	30
31	.26752	.96355	.28429	.95874	.30098	.95363	.31758	.94823	.33408	.94254	29
32	.26780	.96347	.28457	.95865	.30126	.95354	.31786	.94814	.33436	.94245	28
33	.26808	.96340	.28485	.95857	.30154	.95345	.31813	.94805	.33463	.94235	27
34	.26836	.96332	.28513	.95849	.30182	.95337	.31841	.94795	.33490	.94225	26
35	.26864	.96324	.28541	.95841	.30209	.95328	.31868	.94786	.33518	.94215	25
36	.26892	.96316	.28569	.95832	.30237	.95319	.31896	.94777	.33545	.94206	24
37	.26920	.96308	.28597	.95824	.30265	.95310	.31923	.94768	.33573	.94196	23
38	.26948	.96301	.28625	.95816	.30292	.95301	.31951	.94758	.33600	.94186	22
39	.26976	.96293	.28652	.95807	.30320	.95293	.31979	.94749	.33627	.94176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	20
41	.27032	.96277	.28708	.95791	.30376	.95275	.32034	.94730	.33682	.94157	19
42	.27060	.96269	.28736	.95782	.30403	.95266	.32061	.94721	.33710	.94147	18
43	.27088	.96261	.28764	.95774	.30431	.95257	.32089	.94712	.33737	.94137	17
44	.27116	.96253	.28792	.95766	.30459	.95248	.32116	.94702	.33764	.94127	16
45	.27144	.96246	.28820	.95757	.30486	.95240	.32144	.94693	.33792	.94118	15
46	.27172	.96238	.28847	.95749	.30514	.95231	.32171	.94684	.33819	.94108	14
47	.27200	.96230	.28875	.95740	.30542	.95222	.32199	.94674	.33846	.94098	13
48	.27228	.96222	.28903	.95732	.30570	.95213	.32227	.94665	.33874	.94088	12
49	.27256	.96214	.28931	.95724	.30597	.95204	.32254	.94656	.33901	.94078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	10
51	.27312	.96198	.28987	.95707	.30653	.95186	.32309	.94637	.33956	.94058	9
52	.27340	.96190	.29015	.95698	.30680	.95177	.32337	.94627	.33983	.94049	8
53	.27368	.96182	.29042	.95690	.30708	.95168	.32364	.94618	.34011	.94039	7
54	.27396	.96174	.29070	.95681	.30736	.95159	.32392	.94609	.34038	.94029	6
55	.27424	.96166	.29098	.95673	.30763	.95150	.32419	.94599	.34065	.94019	5
56	.27452	.96158	.29126	.95664	.30791	.95142	.32447	.94590	.34093	.94009	4
57	.27480	.96150	.29154	.95656	.30819	.95133	.32474	.94580	.34120	.93999	3
58	.27508	.96142	.29182	.95647	.30846	.95124	.32502	.94571	.34147	.93989	2
59	.27536	.96134	.29209	.95639	.30874	.95115	.32529	.94561	.34175	.93979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	0
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	′
	74°		73°		72°		71°		70°		

TABLE V.

	20°		21°		22°		23°		24°		
′	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	·34202	·93969	·35837	·93358	·37461	·92718	·39073	·92050	·40674	·91355	60
1	·34229	·93959	·35864	·93348	·37488	·92707	·39100	·92039	·40700	·91343	59
2	·34257	·93949	·35891	·93337	·37515	·92697	·39127	·92028	·40727	·91331	58
3	·34284	·93939	·35918	·93327	·37542	·92686	·39153	·92016	·40753	·91319	57
4	·34311	·93929	·35945	·93316	·37569	·92675	·39180	·92005	·40780	·91307	56
5	·34339	·93919	·35973	·93306	·37595	·92664	·39207	·91994	·40806	·91295	55
6	·34366	·93909	·36000	·93295	·37622	·92653	·39234	·91982	·40833	·91283	54
7	·34393	·93899	·36027	·93285	·37649	·92642	·39260	·91971	·40860	·91272	53
8	·34421	·93889	·36054	·93274	·37676	·92631	·39287	·91959	·40886	·91260	52
9	·34448	·93879	·36081	·93264	·37703	·92620	·39314	·91948	·40913	·91248	51
10	·34475	·93869	·36108	·93253	·37730	·92609	·39341	·91936	·40939	·91236	50
11	·34503	·93859	·36135	·93243	·37757	·92598	·39367	·91925	·40966	·91224	49
12	·34530	·93849	·36162	·93232	·37784	·92587	·39394	·91914	·40992	·91212	48
13	·34557	·93839	·36190	·93222	·37811	·92576	·39421	·91902	·41019	·91200	47
14	·34584	·93829	·36217	·93211	·37838	·92565	·39448	·91891	·41045	·91188	46
15	·34612	·93819	·36244	·93201	·37865	·92554	·39474	·91879	·41072	·91176	45
16	·34639	·93809	·36271	·93190	·37892	·92543	·39501	·91868	·41098	·91164	44
17	·34666	·93799	·36298	·93180	·37919	·92532	·39528	·91856	·41125	·91152	43
18	·34694	·93789	·36325	·93169	·37946	·92521	·39555	·91845	·41151	·91140	42
19	·34721	·93779	·36352	·93159	·37973	·92510	·39581	·91833	·41178	·91128	41
20	·34748	·93769	·36379	·93148	·37999	·92499	·39608	·91822	·41204	·91116	40
21	·34775	·93759	·36406	·93137	·38026	·92488	·39635	·91810	·41231	·91104	39
22	·34803	·93748	·36434	·93127	·38053	·92477	·39661	·91799	·41257	·91092	38
23	·34830	·93738	·36461	·93116	·38080	·92466	·39688	·91787	·41284	·91080	37
24	·34857	·93728	·36488	·93106	·38107	·92455	·39715	·91775	·41310	·91068	36
25	·34884	·93718	·36515	·93095	·38134	·92444	·39741	·91764	·41337	·91056	35
26	·34912	·93708	·36542	·93084	·38161	·92432	·39768	·91752	·41363	·91044	34
27	·34939	·93698	·36569	·93074	·38188	·92421	·39795	·91741	·41390	·91032	33
28	·34966	·93688	·36596	·93063	·38215	·92410	·39822	·91729	·41416	·91020	32
29	·34993	·93677	·36623	·93052	·38241	·92399	·39848	·91718	·41443	·91008	31
30	·35021	·93667	·36650	·93042	·38268	·92388	·39875	·91706	·41469	·90996	30
31	·35048	·93657	·36677	·93031	·38295	·92377	·39902	·91694	·41496	·90984	29
32	·35075	·93647	·36704	·93020	·38322	·92366	·39928	·91683	·41522	·90972	28
33	·35102	·93637	·36731	·93010	·38349	·92355	·39955	·91671	·41549	·90960	27
34	·35130	·93626	·36758	·92999	·38376	·92343	·39982	·91660	·41575	·90948	26
35	·35157	·93616	·36785	·92988	·38403	·92332	·40008	·91648	·41602	·90936	25
36	·35184	·93606	·36812	·92978	·38430	·92321	·40035	·91636	·41628	·90924	24
37	·35211	·93596	·36839	·92967	·38456	·92310	·40062	·91625	·41655	·90911	23
38	·35239	·93585	·36867	·92956	·38483	·92299	·40088	·91613	·41681	·90899	22
39	·35266	·93575	·36894	·92945	·38510	·92287	·40115	·91601	·41707	·90887	21
40	·35293	·93565	·36921	·92935	·38537	·92276	·40141	·91590	·41734	·90875	20
41	·35320	·93555	·36948	·92924	·38564	·92265	·40168	·91578	·41760	·90863	19
42	·35347	·93544	·36975	·92913	·38591	·92254	·40195	·91566	·41787	·90851	18
43	·35375	·93534	·37002	·92902	·38617	·92243	·40221	·91555	·41813	·90839	17
44	·35402	·93524	·37029	·92892	·38644	·92231	·40248	·91543	·41840	·90826	16
45	·35429	·93514	·37056	·92881	·38671	·92220	·40275	·91531	·41866	·90814	15
46	·35456	·93503	·37083	·92870	·38698	·92209	·40301	·91519	·41892	·90802	14
47	·35484	·93493	·37110	·92859	·38725	·92198	·40328	·91508	·41919	·90790	13
48	·35511	·93483	·37137	·92849	·38752	·92186	·40355	·91496	·41945	·90778	12
49	·35538	·93472	·37164	·92838	·38778	·92175	·40381	·91484	·41972	·90766	11
50	·35565	·93462	·37191	·92827	·38805	·92164	·40408	·91472	·41998	·90753	10
51	·35592	·93452	·37218	·92816	·38832	·92152	·40434	·91461	·42024	·90741	9
52	·35619	·93441	·37245	·92805	·38859	·92141	·40461	·91449	·42051	·90729	8
53	·35647	·93431	·37272	·92794	·38886	·92130	·40488	·91437	·42077	·90717	7
54	·35674	·93420	·37299	·92784	·38912	·92119	·40514	·91425	·42104	·90704	6
55	·35701	·93410	·37326	·92773	·38939	·92107	·40541	·91414	·42130	·90692	5
56	·35728	·93400	·37353	·92762	·38966	·92096	·40567	·91402	·42156	·90680	4
57	·35755	·93389	·37380	·92751	·38993	·92085	·40594	·91390	·42183	·90668	3
58	·35782	·93379	·37407	·92740	·39020	·92073	·40621	·91378	·42209	·90655	2
59	·35810	·93368	·37434	·92729	·39046	·92062	·40647	·91366	·42235	·90643	1
60	·35837	·93358	·37461	·92718	·39073	·92050	·40674	·91355	·42262	·90631	0
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	′
	69°		68°		67°		66°		65°		

	25°		26°		27°		28°		29°		
′	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	.42262	.90631	.43837	.89879	45399	.89101	.46947	.88295	.48481	.87462	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	.48506	.87448	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	.48532	.87434	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	.48557	.87420	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	.48583	.87406	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	.48608	.87391	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	.48634	.87377	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	.48659	.87363	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	.48684	.87349	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	.48710	.87335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	.48761	.87306	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	.48786	.87292	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	.48811	.87278	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	.48837	.87264	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	.48862	.87250	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	.48888	.87235	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	.48913	.87221	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	.48938	.87207	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	.48964	.87193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	.49014	.87164	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	.49040	.87150	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	.49065	.87136	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	.49090	.87121	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	.49116	.87107	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	.49141	.87093	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	.49166	.87079	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	.49192	.87064	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	.49217	.87050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	.49268	.87021	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	.49293	.87007	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	.49318	.86993	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	.49344	.86978	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	.49369	.86964	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	.49394	.86949	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	.49419	.86935	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	.49445	.86921	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	.49470	.86906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	.49521	.86878	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	.49546	.86863	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	.49571	.86849	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	.49596	.86834	16
45	.43445	.90070	45010	.89298	.46561	.88499	.48099	.87673	.49622	.86820	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	.49647	.86805	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	.49672	.86791	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	.49697	.86777	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	.49723	.86762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	.49773	.86733	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	.49798	.86719	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	.49824	.86704	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	.49849	.86690	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	.49874	.86675	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	.49899	.86661	4
57	43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	.49924	.86646	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	.49950	.86632	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	.49975	.86617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	0
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	′
	64°		63°		62°		61°		60°		

TABLE V.

	30°		31°		32°		33°		34°		
′	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	60
1	.50025	.86588	.51529	.85702	.53017	.84789	.54488	.83851	.55943	.82887	59
2	.50050	.86573	.51554	.85687	.53041	.84774	.54513	.83835	.55968	.82871	58
3	.50076	.86559	.51579	.85672	.53066	.84759	.54537	.83819	.55992	.82855	57
4	.50101	.86544	.51604	.85657	.53091	.84743	.54561	.83804	.56016	.82839	56
5	.50126	.86530	.51628	.85642	.53115	.84728	.54586	.83788	.56040	.82822	55
6	.50151	.86515	.51653	.85627	.53140	.84712	.54610	.83772	.56064	.82806	54
7	.50176	.86501	.51678	.85612	.53164	.84697	.54635	.83756	.56088	.82790	53
8	.50201	.86486	.51703	.85597	.53189	.84681	.54659	.83740	.56112	.82773	52
9	.50227	.86471	.51728	.85582	.53214	.84666	.54683	.83724	.56136	.82757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	50
11	.50277	.86442	.51778	.85551	.53263	.84635	.54732	.83692	.56184	.82724	49
12	.50302	.86427	.51803	.85536	.53288	.84619	.54756	.83676	.56208	.82708	48
13	.50327	.86413	.51828	.85521	.53312	.84604	.54781	.83660	.56232	.82692	47
14	.50352	.86398	.51852	.85506	.53337	.84588	.54805	.83645	.56256	.82675	46
15	.50377	.86384	.51877	.85491	.53361	.84573	.54829	.83629	.56280	.82659	45
16	.50403	.86369	.51902	.85476	.53386	.84557	.54854	.83613	.56305	.82643	44
17	.50428	.86354	.51927	.85461	.53411	.84542	.54878	.83597	.56329	.82626	43
18	.50453	.86340	.51952	.85446	.53435	.84526	.54902	.83581	.56353	.82610	42
19	.50478	.86325	.51977	.85431	.53460	.84511	.54927	.83565	.56377	.82593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	40
21	.50528	.86295	.52026	.85401	.53509	.84480	.54975	.83533	.56425	.82561	39
22	.50553	.86281	.52051	.85385	.53534	.84464	.54999	.83517	.56449	.82544	38
23	.50578	.86266	.52076	.85370	.53558	.84448	.55024	.83501	.56473	.82528	37
24	.50603	.86251	.52101	.85355	.53583	.84433	.55048	.83485	.56497	.82511	36
25	.50628	.86237	.52126	.85340	.53607	.84417	.55072	.83469	.56521	.82495	35
26	.50654	.86222	.52151	.85325	.53632	.84402	.55097	.83453	.56545	.82478	34
27	.50679	.86207	.52175	.85310	.53656	.84386	.55121	.83437	.56569	.82462	33
28	.50704	.86192	.52200	.85294	.53681	.84370	.55145	.83421	.56593	.82446	32
29	.50729	.86178	.52225	.85279	.53705	.84355	.55169	.83405	.56617	.82429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	30
31	.50779	.86148	.52275	.85249	.53754	.84324	.55218	.83373	.56665	.82396	29
32	.50804	.86133	.52299	.85234	.53779	.84308	.55242	.83356	.56689	.82380	28
33	.50829	.86119	.52324	.85218	.53804	.84292	.55266	.83340	.56713	.82363	27
34	.50854	.86104	.52349	.85203	.53828	.84277	.55291	.83324	.56736	.82347	26
35	.50879	.86089	.52374	.85188	.53853	.84261	.55315	.83308	.56760	.82330	25
36	.50904	.86074	.52399	.85173	.53877	.84245	.55339	.83292	.56784	.82314	24
37	.50929	.86059	.52423	.85157	.53902	.84230	.55363	.83276	.56808	.82297	23
38	.50954	.86045	.52448	.85142	.53926	.84214	.55388	.83260	.56832	.82281	22
39	.50979	.86030	.52473	.85127	.53951	.84198	.55412	.83244	.56856	.82264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	20
41	.51029	.86000	.52522	.85096	.54000	.84167	.55460	.83212	.56904	.82231	19
42	.51054	.85985	.52547	.85081	.54024	.84151	.55484	.83195	.56928	.82214	18
43	.51079	.85970	.52572	.85066	.54049	.84135	.55509	.83179	.56952	.82198	17
44	.51104	.85956	.52597	.85051	.54073	.84120	.55533	.83163	.56976	.82181	16
45	.51129	.85941	.52621	.85035	.54097	.84104	.55557	.83147	.57000	.82165	15
46	.51154	.85926	.52646	.85020	.54122	.84088	.55581	.83131	.57024	.82148	14
47	.51179	.85911	.52671	.85005	.54146	.84072	.55605	.83115	.57047	.82132	13
48	.51204	.85896	.52696	.84989	.54171	.84057	.55630	.83098	.57071	.82115	12
49	.51229	.85881	.52720	.84974	.54195	.84041	.55654	.83082	.57095	.82098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	10
51	.51279	.85851	.52770	.84943	.54244	.84009	.55702	.83050	.57143	.82065	9
52	.51304	.85836	.52794	.84928	.54269	.83994	.55726	.83034	.57167	.82048	8
53	.51329	.85821	.52819	.84913	.54293	.83978	.55750	.83017	.57191	.82032	7
54	.51354	.85806	.52844	.84897	.54317	.83962	.55775	.83001	.57215	.82015	6
55	.51379	.85792	.52869	.84882	.54342	.83946	.55799	.82985	.57238	.81999	5
56	.51404	.85777	.52893	.84866	.54366	.83930	.55823	.82969	.57262	.81982	4
57	.51429	.85762	.52918	.84851	.54391	.83915	.55847	.82953	.57286	.81965	3
58	.51454	.85747	.52943	.84836	.54415	.83899	.55871	.82936	.57310	.81949	2
59	.51479	.85732	.52967	.84820	.54440	.83883	.55895	.82920	.57334	.81932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	0
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	′
	59°		58°		57°		56°		55°		

	35°		36°		37°		38°		39°		
′	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	60
1	.57381	.81899	.58802	.80885	.60205	.79846	.61589	.78783	.62955	.77696	59
2	.57405	.81882	.58826	.80867	.60228	.79829	.61612	.78765	.62977	.77678	58
3	.57429	.81865	.58849	.80850	.60251	.79811	.61635	.78747	.63000	.77660	57
4	.57453	.81848	.58873	.80833	.60274	.79793	.61658	.78729	.63022	.77641	56
5	.57477	.81832	.58896	.80816	.60298	.79776	.61681	.78711	.63045	.77623	55
6	.57501	.81815	.58920	.80799	.60321	.79758	.61704	.78694	.63068	.77605	54
7	.57524	.81798	.58943	.80782	.60344	.79741	.61726	.78676	.63090	.77586	53
8	.57548	.81782	.58967	.80765	.60367	.79723	.61749	.78658	.63113	.77568	52
9	.57572	.81765	.58990	.80748	.60390	.79706	.61772	.78640	.63135	.77550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	50
11	.57619	.81731	.59037	.80713	.60437	.79671	.61818	.78604	.63180	.77513	49
12	.57643	.81714	.59061	.80696	.60460	.79653	.61841	.78586	.63203	.77494	48
13	.57667	.81698	.59084	.80679	.60483	.79635	.61864	.78568	.63225	.77476	47
14	.57691	.81681	.59108	.80662	.60506	.79618	.61887	.78550	.63248	.77458	46
15	.57715	.81664	.59131	.80644	.60529	.79600	.61909	.78532	.63271	.77439	45
16	.57738	.81647	.59154	.80627	.60553	.79583	.61932	.78514	.63293	.77421	44
17	.57762	.81631	.59178	.80610	.60576	.79565	.61955	.78496	.63316	.77402	43
18	.57786	.81614	.59201	.80593	.60599	.79547	.61978	.78478	.63338	.77384	42
19	.57810	.81597	.59225	.80576	.60622	.79530	.62001	.78460	.63361	.77366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	40
21	.57857	.81563	.59272	.80541	.60668	.79494	.62046	.78424	.63406	.77329	39
22	.57881	.81546	.59295	.80524	.60691	.79477	.62069	.78405	.63428	.77310	38
23	.57904	.81530	.59318	.80507	.60714	.79459	.62092	.78387	.63451	.77292	37
24	.57928	.81513	.59342	.80489	.60738	.79441	.62115	.78369	.63473	.77273	36
25	.57952	.81496	.59365	.80472	.60761	.79424	.62138	.78351	.63496	.77255	35
26	.57976	.81479	.59389	.80455	.60784	.79406	.62160	.78333	.63518	.77236	34
27	.57999	.81462	.59412	.80438	.60807	.79388	.62183	.78315	.63540	.77218	33
28	.58023	.81445	.59436	.80420	.60830	.79371	.62206	.78297	.63563	.77199	32
29	.58047	.81428	.59459	.80403	.60853	.79353	.62229	.78279	.63585	.77181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	30
31	.58094	.81395	.59506	.80368	.60899	.79318	.62274	.78243	.63630	.77144	29
32	.58118	.81378	.59529	.80351	.60922	.79300	.62297	.78225	.63653	.77125	28
33	.58141	.81361	.59552	.80334	.60945	.79282	.62320	.78206	.63675	.77107	27
34	.58165	.81344	.59576	.80316	.60968	.79264	.62342	.78188	.63698	.77088	26
35	.58189	.81327	.59599	.80299	.60991	.79247	.62365	.78170	.63720	.77070	25
36	.58212	.81310	.59622	.80282	.61015	.79229	.62388	.78152	.63742	.77051	24
37	.58236	.81293	.59646	.80264	.61038	.79211	.62411	.78134	.63765	.77033	23
38	.58260	.81276	.59669	.80247	.61061	.79193	.62433	.78116	.63787	.77014	22
39	.58283	.81259	.59693	.80230	.61084	.79176	.62456	.78098	.63810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	20
41	.58330	.81225	.59739	.80195	.61130	.79140	.62502	.78061	.63854	.76959	19
42	.58354	.81208	.59763	.80178	.61153	.79122	.62524	.78043	.63877	.76940	18
43	.58378	.81191	.59786	.80160	.61176	.79105	.62547	.78025	.63899	.76921	17
44	.58401	.81174	.59809	.80143	.61199	.79087	.62570	.78007	.63922	.76903	16
45	.58425	.81157	.59832	.80125	.61222	.79069	.62592	.77988	.63944	.76884	15
46	.58449	.81140	.59856	.80108	.61245	.79051	.62615	.77970	.63966	.76866	14
47	.58472	.81123	.59879	.80091	.61268	.79033	.62638	.77952	.63989	.76847	13
48	.58496	.81106	.59902	.80073	.61291	.79016	.62660	.77934	.64011	.76828	12
49	.58519	.81089	.59926	.80056	.61314	.78998	.62683	.77916	.64033	.76810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	10
51	.58567	.81055	.59972	.80021	.61360	.78962	.62728	.77879	.64078	.76772	9
52	.58590	.81038	.59995	.80003	.61383	.78944	.62751	.77861	.64100	.76754	8
53	.58614	.81021	.60019	.79986	.61406	.78926	.62774	.77843	.64123	.76735	7
54	.58637	.81004	.60042	.79968	.61429	.78908	.62796	.77824	.64145	.76717	6
55	.58661	.80987	.60065	.79951	.61451	.78891	.62819	.77806	.64167	.76698	5
56	.58684	.80970	.60089	.79934	.61474	.78873	.62842	.77788	.64190	.76679	4
57	.58708	.80953	.60112	.79916	.61497	.78855	.62864	.77769	.64212	.76661	3
58	.58731	.80936	.60135	.79899	.61520	.78837	.62887	.77751	.64234	.76642	2
59	.58755	.80919	.60158	.79881	.61543	.78819	.62909	.77733	.64256	.76623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	0
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	′

	40°		41°		42°		43°		44°		
′	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	
0	.64279	.76604	.65606	.75471	66913	.74314	.68200	.73135	.69466	.71934	60
1	.64301	.76586	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.64323	.76567	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.64346	.76548	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.64368	.76530	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.64412	.76492	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.64435	.76473	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.64457	.76455	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.64479	.76436	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.64524	.76398	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.64546	.76380	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.64568	.76361	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.64590	.76342	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.64635	.76304	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.64657	.76286	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.64679	.76267	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.64701	.76248	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.64746	.76210	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.64768	.76192	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.64790	.76173	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.64812	.76154	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.64856	.76116	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.64878	.76097	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.64901	.76078	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.64923	.76059	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.64967	.76022	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.64989	.76003	.66306	.74857	.67602	.73688	.68878	.72497	.70132	.71284	28
33	.65011	.75984	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.65033	.75965	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.65077	.75927	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.65100	.75908	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.65122	.75889	.66436	.74741	.67730	.73570	.69004	.72377	.70257	.71162	22
39	.65144	.75870	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.65188	.75832	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.65210	.75813	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.65232	.75794	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.65254	.75775	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.65276	.75756	66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.65298	.75738	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.65320	.75719	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.65342	.75700	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.65364	.75680	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.65408	.75642	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.65430	.75623	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.65452	.75604	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.65474	.75585	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.65518	.75547	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.65540	.75528	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.65562	.75509	.66870	.74353	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.65584	.75490	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	N. cos.	N. sine	′
	49°		48°		47°		46°		45°		

TABLE VI.

ADDITION AND SUBTRACTION LOGARITHMS.

PRECEPTS.

I. *When difference of given logarithms is less than* 2.00.

ADDITION.—Enter table with difference between logarithms as *Arg. A*, and take out *B*.

Add *B* to subtracted logarithm.

SUBTRACTION.—Subtract lesser from greater logarithm; enter with the difference as *B*, and take out *A*.

Add *A* to the subtracted logarithm.

II. *When difference of given logarithms exceeds* 2.00.

Subtract lesser from greater.

ADDITION.—Enter table with difference as *Arg. A*, take out *B—A* and add it to the greater logarithm.

SUBTRACTION.—Enter column *B* with difference of logarithms; take out *B—A*, and subtract it from greater logarithm.

A.	B.	0	1	2	3	4	5	6	7	8	9
5.	0.00	000	001	001	001	001	001	002	002	003	003
6.0		004	004	00$\bar{5}$	00$\bar{5}$	00$\bar{5}$	00$\bar{5}$	00$\bar{5}$	005	005	005
6.1		005	006	006	006	006	006	006	006	007	007
6.2		007	007	007	007	008	008	008	008	008	008
6.3		009	009	009	009	010	010	010	010	010	011
6.4		011	011	011	012	012	012	013	013	013	013
6.5		014	014	014	01$\bar{5}$	015	015	016	016	017	017
6.6		017	018	018	019	019	019	020	020	021	021
6.7		022	022	023	023	024	024	02$\bar{5}$	026	026	027
6.8		027	028	029	029	030	031	031	032	033	034
6.9		034	035	036	037	038	039	040	041	041	042
7.0		043	044	045	047	048	049	0$\bar{5}$0	051	052	053
7.1		05$\bar{5}$	056	057	059	060	061	063	064	066	067
7.2		069	070	072	074	075	077	079	081	083	08$\bar{5}$
7.3		087	089	091	093	09$\bar{5}$	097	099	102	104	106
7.4		109	111	114	117	119	122	125	128	131	134
7.5		137	140	144	147	150	154	157	161	16$\bar{5}$	169
7.6		173	177	181	18$\bar{5}$	189	194	198	203	207	212
7.7		217	222	227	233	238	244	249	25$\bar{5}$	261	267
7.8		273	280	286	293	299	306	313	321	328	336
7.9		344	352	360	368	377	385	394	403	413	422
8.0		432	442	452	463	474	48$\bar{5}$	496	507	519	531
A.	B.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	3	4	5	6
1	0.3	0.4	0.5	0.6
2	0.6	0.8	1.0	1.2
3	0.9	1.2	1.5	1.8
4	1.2	1.6	2.0	2.4
5	1.5	2.0	2.5	3.0
6	1.8	2.4	3.0	3.6
7	2.1	2.8	3.5	4.2
8	2.4	3.2	4.0	4.8
9	2.7	3.6	4.5	5.4

	7	8	9	10
1	0.7	0.8	0.9	1.0
2	1.4	1.6	1.8	2.0
3	2.1	2.4	2.7	3.0
4	2.8	3.2	3.6	4.0
5	3.5	4.0	4.5	5.0
6	4.2	4.8	5.4	6.0
7	4.9	5.6	6.3	7.0
8	5.6	6.4	7.2	8.0
9	6.3	7.2	8.1	9.0

Prop. Pts.

Add. $\begin{cases} \log b - \log a = A. \\ \log (a + b) = \log a + B. \end{cases}$ **Sub.** $\begin{cases} \log a - \log b = B. \\ \log (a - b) = \log b + A. \end{cases}$

A.	B.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
8.00	0.00	432	433	434	435	436	437	438	439	440	441	
8.01		442	443	444	445	446	447	448	449	450	451	
8.02		452	453	454	456	457	458	459	460	461	462	
8.03		463	464	465	466	467	468	469	470	471	473	
8.04		474	475̄	476	477	478	479	480	481	482	483	
8.05		485̄	486	487	488	489	490	491	492	494	495̄	
8.06		496	497	498	499	500	502	503	504	505̄	506	
8.07		507	508	510	511	512	513	514	515	517	518	
8.08		519	520	521	523	524	525̄	526	527	529	530	
8.09		531	532	533	535̄	536	537	538	540	541	542	
8.10		543	545̄	546	547	548	55̄0	551	552	553	555̄	
8.11		556	557	558	560	561	562	564	565̄	566	567	
8.12		569	570	571	573	574	575	577	578	579	581	
8.13		582	583	585̄	586	587	589	590	591	593	594	
8.14		595	597	598	599	601	602	604	605	606	608	
8.15		609	611	612	613	615̄	616	618	619	620	622	
8.16		623	625̄	626	628	629	630	632	633	635̄	636	
8.17		638	639	641	642	644	645̄	646	648	649	651	
8.18		652	654	655	657	658	660	661	663	664	666	
8.19		667	669	671	672	674	675	677	678	680	681	
8.20		683	684	686	688	689	691	692	694	696	697	
8.21		699	700	702	703	705	707	708	710	712	713	
8.22		715̄	716	718	720	721	723	725̄	726	728	730	
8.23		731	733	735̄	736	738	740	741	743	745̄	747	
8.24		748	75̄0	752	753	755	757	759	760	762	764	
8.25		766	767	769	771	773	774	776	778	780	781	
8.26		783	785̄	787	789	790	792	794	796	798	799	
8.27		801	803	805̄	807	809	810	812	814	816	818	
8.28		820	822	823	825	827	829	831	833	835̄	837	
8.29		839	841	842	844	846	848	850	852	854	856	
8.30		858	860	862	864	866	868	870	872	874	876	
8.31		878	880	882	884	886	888	890	892	894	896	
8.32		898	900	902	904	906	908	910	912	915̄	917	
8.33		919	921	923	925	927	929	931	933	936	938	
8.34		940	942	944	946	948	951	953	955̄	957	959	
8.35		962	964	966	968	970	973	975̄	977	979	981	
8.36		984	986	988	990	993	995̄	997	999	*002	*004	
8.37	0.01	006	009	011	013	016	018	020	022	025̄	027	
8.38		030	032	034	037	039	041	044	046	048	051	
8.39		053	056	058	060	063	065	068	070	073	075̄	
8.40		077	080	082	085̄	087	090	092	095̄	097	100	
8.41		102	105̄	107	110	112	115̄	117	120	122	125̄	
8.42		128	130	133	135	138	140	143	146	148	151	
8.43		153	156	159	161	164	167	169	172	175̄	177	
8.44		180	183	185	188	191	193	196	199	202	204	
8.45		207	210	213	215	218	221	224	226	229	232	
8.46		235̄	238	240	243	246	249	252	255̄	257	260	
8.47		263	266	269	272	275̄	278	280	283	286	289	
8.48		292	295	298	301	304	307	310	313	316	319	
8.49		322	325̄	328	331	334	337	340	343	346	349	
8.50		352	355	358	361	364	368	371	374	377	380	
A.	B.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.

	2
1	0.2
2	0.4
3	0.6
4	0.8
5	1.0
6	1.2
7	1.4
8	1.6
9	1.8

	3
1	0.3
2	0.6
3	0.9
4	1.2
5	1.5
6	1.8
7	2.1
8	2.4
9	2.7

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

ADD. $\begin{cases} \log b - \log a = A. \\ \log(a + b) = \log a + B. \end{cases}$ SUB. $\begin{cases} \log a - \log b = B. \\ \log(a - b) = \log b + A. \end{cases}$

A.	B.	0	1	2	3	4	5	6	7	8	9
8.50	0.01	352	355	358	361	364	368	371	374	377	380
8.51		383	386	389	393	396	399	402	405	408	412
8.52		41$\bar{5}$	418	421	424	428	431	434	437	441	444
8.53		447	450	454	457	460	464	467	470	474	477
8.54		480	484	487	490	494	497	501	504	507	511
8.55		514	518	521	52$\bar{5}$	528	531	53$\bar{5}$	538	542	545
8.56		549	552	556	559	563	566	570	574	577	581
8.57		584	588	591	595	599	602	606	610	613	617
8.58		621	624	628	632	635	639	643	646	650	654
8.59		658	661	665	669	673	676	680	684	688	692
8.60		695	699	703	707	711	71$\bar{5}$	719	722	726	730
8.61		734	738	742	746	7$\bar{5}$0	754	758	762	766	770
8.62		774	778	782	786	790	794	798	802	806	810
8.63		814	818	822	827	831	83$\bar{5}$	839	843	847	851
8.64		856	860	864	868	872	877	881	885	889	894
8.65		898	902	906	911	915	919	924	928	932	937
8.66		941	945	9$\bar{5}$0	954	959	963	967	972	976	981
8.67		985	990	994	999	*003	*008	*012	*017	*021	*026
8.68	0.02	030	035	040	044	049	053	058	063	067	072
8.69		077	081	086	091	095	100	10$\bar{5}$	110	114	119
8.70		124	129	133	138	143	148	153	158	162	167
8.71		172	177	182	187	192	197	202	207	211	216
8.72		221	226	231	236	241	246	252	257	262	267
8.73		272	277	282	287	292	297	303	308	313	318
8.74		323	329	334	339	344	3$\bar{5}$0	35$\bar{5}$	360	365	371
8.75		376	381	387	392	397	403	408	414	419	424
8.76		430	435	441	446	452	457	463	468	474	479
8.77		48$\bar{5}$	490	496	502	507	513	518	524	530	535
8.78		541	547	552	558	564	570	575	581	587	593
8.79		599	604	610	616	622	628	634	639	645	651
8.80		657	663	669	675	681	687	693	699	705	711
8.81		717	723	729	735	742	748	754	760	766	772
8.82		779	78$\bar{5}$	791	797	803	810	816	822	829	83$\bar{5}$
8.83		841	848	854	860	867	873	879	886	892	899
8.84		905	912	918	92$\bar{5}$	931	938	944	951	957	964
8.85		971	977	984	991	997	*004	*011	*017	*024	*031
8.86	0.03	037	044	051	058	06$\bar{5}$	071	078	085	092	099
8.87		106	113	120	126	133	140	147	154	161	168
8.88		175	183	190	197	204	211	218	225	232	240
8.89		247	254	261	268	276	283	290	298	30$\bar{5}$	312
8.90		320	327	334	342	349	357	364	371	379	386
8.91		394	401	409	417	424	432	439	447	45$\bar{5}$	462
8.92		470	478	485	493	501	509	516	524	532	540
8.93		548	555	563	571	579	587	59$\bar{5}$	603	611	619
8.94		627	63$\bar{5}$	643	651	659	667	675	683	691	700
8.95		708	716	724	732	741	749	757	765	774	782
8.96		790	799	807	816	824	832	841	849	858	866
8.97		87$\bar{5}$	883	892	901	909	918	926	935	944	953
8.98		961	970	979	987	996	*005	*014	*023	*032	*040
8.99	0.04	049	058	067	076	085	094	103	112	121	130
9.00		139	148	157	167	176	18$\bar{5}$	194	203	213	222
A.	**B.**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Pts.

	3	4
1	0.3	0.4
2	0.6	0.8
3	0.9	1.2
4	1.2	1.6
5	1.5	2.0
6	1.8	2.4
7	2.1	2.8
8	2.4	3.2
9	2.7	3.6

	5	6
1	0.5	0.6
2	1.0	1.2
3	1.5	1.8
4	2.0	2.4
5	2.5	3.0
6	3.0	3.6
7	3.5	4.2
8	4.0	4.8
9	4.5	5.4

	7	8
1	0.7	0.8
2	1.4	1.6
3	2.1	2.4
4	2.8	3.2
5	3.5	4.0
6	4.2	4.8
7	4.9	5.6
8	5.6	6.4
9	6.3	7.2

	9	10
1	0.9	1.0
2	1.8	2.0
3	2.7	3.0
4	3.6	4.0
5	4.5	5.0
6	5.4	6.0
7	6.3	7.0
8	7.2	8.0
9	8.1	9.0

Prop. Pts.

Add. $\left\{\begin{array}{l}\log b - \log a = A.\\ \log(a+b) = \log a + B.\end{array}\right.$ **Sub.** $\left\{\begin{array}{l}\log a - \log b = B.\\ \log(a-b) = \log b + A.\end{array}\right.$

A.	B.	0	1	2	3	4	5	6	7	8	9
9.00	0.04	139	148	157	167	176	185̅	194	203	213	222
9.01		231	240	25̅0	259	268	278	287	297	306	315
9.02		325̅	334	344	353	363	373	382	392	401	411
9.03		421	430	440	45̅0	460	469	479	489	499	509
9.04		519	528	538	548	558	568	578	588	598	608
9.05		618	628	639	649	659	669	679	689	700	710
9.06		720	731	741	751	762	772	782	793	803	814
9.07		824	835̅	845	856	867	877	888	898	909	920
9.08		931	941	952	963	974	985̅	995	*006	*017	*028
9.09	0.05	039	05̅0	061	072	083	094	105	116	127	139
9.10		15̅0	161	172	183	195̅	206	217	229	240	251
9.11		263	274	286	297	308	320	332	343	355̅	366
9.12		378	390	401	413	425̅	436	448	460	472	484
9.13		496	508	519	531	543	555	567	579	591	604
9.14		616	628	640	652	664	677	689	701	714	726
9.15		738	751	763	775	788	800	813	825	838	851
9.16		863	876	889	901	914	927	939	952	965	978
9.17		991	*004	*017	*030	*043	*056	*069	*082	*095̅	*108
9.18	0.06	121	134	147	161	174	187	200	214	227	240
9.19		254	267	281	294	308	321	335̅	348	362	376
9.20		389	403	417	430	444	458	472	486	5̅00	513
9.21		527	541	555	569	583	597	612	626	640	654
9.22		668	683	697	711	725	740	754	769	783	798
9.23		812	827	841	856	870	885	900	914	929	944
9.24		959	973	988	*003	*018	*033	*048	*063	*078	*093
9.25	0.07	108	123	138	154	169	184	199	215̅	230	245
9.26		261	276	291	307	322	338	354	369	385̅	400
9.27		416	432	448	463	479	495̅	511	527	543	559
9.28		575̅	591	607	623	639	655	671	687	704	720
9.29		736	753	769	785	802	818	835̅	851	868	884
9.30		901	918	934	951	968	985̅	*001	*018	*035	*052
9.31	0.08	069	086	103	120	137	154	171	188	206	223
9.32		240	257	275̅	292	309	327	344	362	379	397
9.33		415̅	432	45̅0	468	485	503	521	539	557	574
9.34		592	610	628	646	664	683	701	719	737	755
9.35		774	792	810	829	847	865	884	902	921	940
9.36		958	977	996	*014	*033	*052	*071	*090	*108	*127
9.37	0.09	146	165	184	204	223	242	261	280	299	319
9.38		338	357	377	396	416	435	455̅	474	494	514
9.39		533	553	573	593	612	632	652	672	692	712
9.40		732	752	773	793	813	833	853	874	894	914
9.41		935̅	955	976	996	*017	*038	*058	*079	*100	*120
9.42	0.10	141	162	183	204	225̅	246	267	288	309	330
9.43		351	373	394	415	437	458	479	501	522	544
9.44		565	587	609	630	652	674	696	718	739	761
9.45		783	805	827	849	872	894	916	938	960	983
9.46	0.11	005	028	050	073	095	118	140	163	186	208
9.47		231	254	277	300	323	345	368	392	415̅	438
9.48		461	484	507	531	554	577	601	624	648	671
9.49		695̅	719	742	766	790	814	837	861	885	909
9.50		933	957	981	*005	*030	*054	*078	*102	*127	*151
A.	B.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	9	10	11
1	0.9	1.0	1.1
2	1.8	2.0	2.2
3	2.7	3.0	3.3
4	3.6	4.0	4.4
5	4.5	5.0	5.5
6	5.4	6.0	6.6
7	6.3	7.0	7.7
8	7.2	8.0	8.8
9	8.1	9.0	9.9

	12	13	14
1	1.2	1.3	1.4
2	2.4	2.6	2.8
3	3.6	3.9	4.2
4	4.8	5.2	5.6
5	6.0	6.5	7.0
6	7.2	7.8	8.4
7	8.4	9.1	9.8
8	9.6	10.4	11.2
9	10.8	11.7	12.6

	15	16	17
1	1.5	1.6	1.7
2	3.0	3.2	3.4
3	4.5	4.8	5.1
4	6.0	6.4	6.8
5	7.5	8.0	8.5
6	9.0	9.6	10.2
7	10.5	11.2	11.9
8	12.0	12.8	13.6
9	13.5	14.4	15.3

	18	19	20
1	1.8	1.9	2.0
2	3.6	3.8	4.0
3	5.4	5.7	6.0
4	7.2	7.6	8.0
5	9.0	9.5	10.0
6	10.8	11.4	12.0
7	12.6	13.3	14.0
8	14.4	15.2	16.0
9	16.2	17.1	18.0

	21	22	23
1	2.1	2.2	2.3
2	4.2	4.4	4.6
3	6.3	6.6	6.9
4	8.4	8.8	9.2
5	10.5	11.0	11.5
6	12.6	13.2	13.8
7	14.7	15.4	16.1
8	16.8	17.6	18.4
9	18.9	19.8	20.7

	24	25	26
1	2.4	2.5	2.6
2	4.8	5.0	5.2
3	7.2	7.5	7.8
4	9.6	10.0	10.4
5	12.0	12.5	13.0
6	14.4	15.0	15.6
7	16.8	17.5	18.2
8	19.2	20.0	20.8
9	21.6	22.5	23.4

Prop. Pts.

Add. $\left\{\begin{array}{l}\log b - \log a = A.\\ \log(a+b) = \log a + B.\end{array}\right.$ **Sub.** $\left\{\begin{array}{l}\log a - \log b = B.\\ \log(a-b) = \log b + A.\end{array}\right.$

A.	B.	0	1	2	3	4	5	6	7	8	9
9.50	0.11	933	957	981	*005	*030	*054	*078	*102	*127	*151
9.51	0.12	175	200	224	249	274	298	323	348	372	397
9.52		422	447	472	497	522	547	572	597	622	648
9.53		673	698	724	749	$77\bar{5}$	800	826	851	877	903
9.54		928	954	980	*006	*032	*058	*084	*110	*136	*162
9.55	0.13	188	214	240	267	293	319	346	372	399	425
9.56		452	479	505	532	559	586	613	640	667	694
9.57		721	748	$77\bar{5}$	802	829	857	884	911	939	966
9.58		994	*021	*049	*077	*104	*132	*160	*188	*216	*244
9.59	0.14	272	300	328	356	384	412	441	469	497	526
9.60		554	583	611	640	668	697	726	$75\bar{5}$	783	812
9.61		841	870	899	928	957	986	*016	*$04\bar{5}$	*074	*104
9.62	0.15	133	162	192	221	251	281	310	340	370	400
9.63		430	460	489	520	$5\bar{5}0$	580	610	640	670	701
9.64		731	761	792	822	853	884	914	$94\bar{5}$	976	*007
9.65	0.16	037	068	099	130	161	192	224	$25\bar{5}$	286	317
9.66		349	380	411	443	474	506	538	569	601	633
9.67		$66\bar{5}$	697	729	761	793	$82\bar{5}$	857	889	921	954
9.68		986	*018	*051	*083	*116	*148	*181	*214	*247	*279
9.69	0.17	312	345	378	411	444	477	510	544	577	610
9.70		643	677	710	744	777	811	$84\bar{5}$	878	912	946
9.71		980	*014	*048	*082	*116	*150	*184	*218	*253	*287
9.72	0.18	322	356	390	$42\bar{5}$	460	494	529	564	599	633
9.73		668	703	738	773	808	844	879	914	949	$98\bar{5}$
9.74	0.19	020	056	091	127	163	198	234	270	306	342
9.75		378	414	$4\bar{5}0$	486	522	558	$59\bar{5}$	631	667	704
9.76		740	777	813	850	887	923	960	997	*034	*071
9.77	0.20	108	145	182	220	257	294	331	369	406	444
9.78		481	519	557	594	632	670	708	746	784	822
9.79		860	898	937	$97\bar{5}$	*013	*052	*090	*128	*167	*206
9.80	0.21	244	283	322	361	399	438	477	516	556	$59\bar{5}$
9.81		634	673	712	752	791	831	870	910	949	989
9.82	0.22	029	069	109	149	189	229	269	309	349	389
9.83		430	470	510	551	591	632	673	713	754	$79\bar{5}$
9.84		836	877	918	959	*000	*041	*082	*123	*$16\bar{5}$	*206
9.85	0.23	247	289	330	372	414	455	497	539	581	623
9.86		$66\bar{5}$	707	749	791	833	875	918	960	*003	*045
9.87	0.24	088	130	173	216	258	301	344	387	430	473
9.88		516	559	603	646	689	733	776	819	863	907
9.89		950	994	*038	*082	*126	*170	*214	*258	*302	*346
9.90	0.25	390	434	479	523	568	612	657	701	746	791
9.91		836	881	926	970	*016	*061	*106	*151	*196	*242
9.92	0.26	287	332	378	423	469	$51\bar{5}$	560	606	652	698
9.93		744	790	836	882	928	974	*021	*067	*114	*160
9.94	0.27	207	253	300	346	393	440	487	534	581	628
9.95		$67\bar{5}$	722	769	817	864	911	959	*006	*054	*101
9.96	0.28	149	197	$24\bar{5}$	292	340	388	436	484	532	581
9.97		629	677	726	774	822	871	920	968	*017	*066
9.98	0.29	$11\bar{5}$	163	212	261	310	359	409	458	507	556
9.99		606	655	$70\bar{5}$	754	804	854	903	953	*003	*053
0.00	0.30	103	153	203	253	303	354	404	454	$50\bar{5}$	555
A.	B.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	27	28	29	30
1	2.7	2.8	2.9	3.0
2	5.4	5.6	5.8	6.0
3	8.1	8.4	8.7	9.0
4	10.8	11.2	11.6	12.0
5	13.5	14.0	14.5	15.0
6	16.2	16.8	17.4	18.0
7	18.9	19.6	20.3	21.0
8	21.6	22.4	23.2	24.0
9	24.3	25.2	26.1	27.0

	31	32	33	34
1	3.1	3.2	3.3	3.4
2	6.2	6.4	6.6	6.8
3	9.3	9.6	9.9	10.2
4	12.4	12.8	13.2	13.6
5	15.5	16.0	16.5	17.0
6	18.6	19.2	19.8	20.4
7	21.7	22.4	23.1	23.8
8	24.8	25.6	26.4	27.2
9	27.9	28.8	29.7	30.6

	35	36	37	38
1	3.5	3.6	3.7	3.8
2	7.0	7.2	7.4	7.6
3	10.5	10.8	11.1	11.4
4	14.0	14.4	14.8	15.2
5	17.5	18.0	18.5	19.0
6	21.0	21.6	22.2	22.8
7	24.5	25.2	25.9	26.6
8	28.0	28.8	29.6	30.4
9	31.5	32.4	33.3	34.2

	39	40	41	42
1	3.9	4.0	4.1	4.2
2	7.8	8.0	8.2	8.4
3	11.7	12.0	12.3	12.6
4	15.6	16.0	16.4	16.8
5	19.5	20.0	20.5	21.0
6	23.4	24.0	24.6	25.2
7	27.3	28.0	28.7	29.4
8	31.2	32.0	32.8	33.6
9	35.1	36.0	36.9	37.8

	43	44	45	46
1	4.3	4.4	4.5	4.6
2	8.6	8.8	9.0	9.2
3	12.9	13.2	13.5	13.8
4	17.2	17.6	18.0	18.4
5	21.5	22.0	22.5	23.0
6	25.8	26.4	27.0	27.6
7	30.1	30.8	31.5	32.2
8	34.4	35.2	36.0	36.8
9	38.7	39.6	40.5	41.4

	47	48	49	50
1	4.7	4.8	4.9	5.0
2	9.4	9.6	9.8	10.0
3	14.1	14.4	14.7	15.0
4	18.8	19.2	19.6	20.0
5	23.5	24.0	24.5	25.0
6	28.2	28.8	29.4	30.0
7	32.9	33.6	34.3	35.0
8	37.6	38.4	39.2	40.0
9	42.3	43.2	44.1	45.0

Prop. Pts.

TABLE VI.

Add. $\begin{cases} \log a - \log b = A. \\ \log(a+b) = \log b + B. \end{cases}$ **Sub.** $\begin{cases} \log a - \log b = B. \\ \log(a-b) = \log b + A. \end{cases}$

A.	B.	0	1	2	3	4	5	6	7	8	9
0.00	0.30	103	153	203	253	303	354	404	454	505̄	555
0.01		606	656	707	758	809	859	910	961	*012	*063
0.02	0.31	115̄	166	217	268	320	371	422	474	526	577
0.03		629	681	732	784	836	888	940	992	*045̄	*097
0.04	0.32	149	201	254	306	359	411	464	517	569	622
0.05		675̄	728	781	834	887	940	993	*046	*100	*153
0.06	0.33	207	260	314	367	421	474	528	582	636	690
0.07		744	798	852	906	960	*015̄	*069	*123	*178	*232
0.08	0.34	287	342	396	451	506	561	616	670	726	781
0.09		836	891	946	*001	*057	*112	*168	*223	*279	*334
0.10	0.35	390	446	502	558	614	670	726	782	838	894
0.11		950	*007	*063	*119	*176	*233	*289	*346	*403	*459
0.12	0.36	516	573	630	687	744	801	858	916	973	*030
0.13	0.37	088	145	203	260	318	375	433	491	549	607
0.14		665̄	723	781	839	897	955	*014	*072	*130	*189
0.15	0.38	247	306	365̄	423	482	541	600	659	718	777
0.16		836	895̄	954	*013	*073	*132	*191	*251	*310	*370
0.17	0.39	430	489	549	609	669	729	789	849	909	969
0.18	0.40	029	089	149	210	270	331	391	452	512	573
0.19		634	695̄	756	816	877	938	999	*061	*122	*183
0.20	0.41	244	306	367	428	490	552	613	675̄	737	798
0.21		860	922	984	*046	*108	*170	*232	*294	*357	*419
0.22	0.42	481	544	606	669	731	794	857	920	982	*045
0.23	0.43	108	171	234	297	360	423	487	550	613	677
0.24		740	804	867	931	995̄	*058	*122	*186	*25̄0	*314
0.25	0.44	378	442	506	570	634	698	763	827	891	956
0.26	0.45	020	085̄	149	214	279	344	408	473	538	603
0.27		668	733	799	864	929	994	*060	*125̄	*190	*256
0.28	0.46	322	387	453	518	584	650	716	782	848	914
0.29		980	*046	*112	*178	*245̄	*311	*377	*444	*510	*577
0.30	0.47	643	710	777	844	910	977	*044	*111	*178	*245
0.31	0.48	312	379	447	514	581	648	716	783	851	918
0.32		986	*054	*121	*189	*257	*325̄	*393	*461	*529	*597
0.33	0.49	665̄	733	801	869	938	*006	*074	*143	*211	*280
0.34	0.50	349	417	486	555̄	624	692	761	830	899	968
0.35	0.51	037	107	176	245̄	314	384	453	522	592	661
0.36		731	801	870	940	*010	*080	*15̄0	*220	*289	*360
0.37	0.52	430	5̄00	570	640	710	781	851	921	992	*062
0.38	0.53	133	204	274	345̄	416	486	557	628	699	770
0.39		841	912	983	*055̄	*126	*197	*268	*340	*411	*483
0.40	0.54	554	626	697	769	841	912	984	*056	*128	*200
0.41	0.55	272	344	416	488	560	632	704	777	849	921
0.42		994	*066	*139	*211	*284	*357	*429	*502	*575̄	*648
0.43	0.56	721	794	867	940	*013	*086	*159	*232	*305	*379
0.44	0.57	452	525	599	672	746	819	893	967	*040	*114
0.45	0.58	188	262	336	410	484	558	632	706	780	854
0.46		928	*003	*077	*151	*226	*300	*375̄	*449	*524	*598
0.47	0.59	673	748	822	897	972	*047	*122	*197	*272	*347
0.48	0.60	422	497	572	648	723	798	874	949	*024	*100
0.49	0.61	175	251	327	402	478	554	630	705	781	857
0.50		933	*009	*085	*161	*237	*314	*390	*466	*542	*619
A.	**B.**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Pts.

	50	51	52	53
1	5.0	5.1	5.2	5.3
2	10.0	10.2	10.4	10.6
3	15.0	15.3	15.6	15.9
4	20.0	20.4	20.8	21.2
5	25.0	25.5	26.0	26.5
6	30.0	30.6	31.2	31.8
7	35.0	35.7	36.4	37.1
8	40.0	40.8	41.6	42.4
9	45.0	45.9	46.8	47.7

	54	55	56	57
1	5.4	5.5	5.6	5.7
2	10.8	11.0	11.2	11.4
3	16.2	16.5	16.8	17.1
4	21.6	22.0	22.4	22.8
5	27.0	27.5	28.0	28.5
6	32.4	33.0	33.6	34.2
7	37.8	38.5	39.2	39.9
8	43.2	44.0	44.8	45.6
9	48.6	49.5	50.4	51.3

	58	59	60	61
1	5.8	5.9	6.0	6.1
2	11.6	11.8	12.0	12.2
3	17.4	17.7	18.0	18.3
4	23.2	23.6	24.0	24.4
5	29.0	29.5	30.0	30.5
6	34.8	35.4	36.0	36.6
7	40.6	41.3	42.0	42.7
8	46.4	47.2	48.0	48.8
9	52.2	53.1	54.0	54.9

	62	63	64	65
1	6.2	6.3	6.4	6.5
2	12.4	12.6	12.8	13.0
3	18.6	18.9	19.2	19.5
4	24.8	25.2	25.6	26.0
5	31.0	31.5	32.0	32.5
6	37.2	37.8	38.4	39.0
7	43.4	44.1	44.8	45.5
8	49.6	50.4	51.2	52.0
9	55.8	56.7	57.6	58.5

	66	67	68	69
1	6.6	6.7	6.8	6.9
2	13.2	13.4	13.6	13.8
3	19.8	20.1	20.4	20.7
4	26.4	26.8	27.2	27.6
5	33.0	33.5	34.0	34.5
6	39.6	40.2	40.8	41.4
7	46.2	46.9	47.6	48.3
8	52.8	53.6	54.4	55.2
9	59.4	60.3	61.2	62.1

	70	71	72	73
1	7.0	7.1	7.2	7.3
2	14.0	14.2	14.4	14.6
3	21.0	21.3	21.6	21.9
4	28.0	28.4	28.8	29.2
5	35.0	35.5	36.0	36.5
6	42.0	42.6	43.2	43.8
7	49.0	49.7	50.4	51.1
8	56.0	56.8	57.6	58.4
9	63.0	63.9	64.8	65.7

Prop. Pts.

Add. $\begin{cases} \log a - \log b = A. \\ \log(a+b) = \log b + B. \end{cases}$ **Sub.** $\begin{cases} \log a - \log b = B. \\ \log(a-b) = \log b + A. \end{cases}$

A.	B.	0	1	2	3	4	5	6	7	8	9
0.50	0.61	933	*009	*085	*161	*237	*314	*390	*466	*542	*619
0.51	0.62	69$\bar{5}$	771	848	924	*001	*077	*154	*231	*307	*384
0.52	0.63	461	538	61$\bar{5}$	692	768	845	923	*000	*077	*154
0.53	0.64	231	308	386	463	540	618	695	773	850	928
0.54	0.65	005	083	160	238	316	394	472	549	627	705
0.55		783	861	939	*018	*096	*174	*252	*330	*409	*487
0.56	0.66	565	644	722	801	879	958	*037	*115	*194	*273
0.57	0.67	351	430	509	588	667	746	82$\bar{5}$	904	983	*062
0.58	0.68	141	220	300	379	458	538	617	696	776	855
0.59		93$\bar{5}$	*014	*094	*174	*253	*333	*413	*493	*573	*652
0.60	0.69	732	812	892	972	*052	*132	*212	*293	*373	*453
0.61	0.70	533	614	694	774	85$\bar{5}$	935	*016	*096	*177	*257
0.62	0.71	338	419	499	580	661	742	823	904	984	*065
0.63	0.72	146	227	308	390	471	552	633	714	796	877
0.64		958	*040	*121	*202	*284	*365	*447	*529	*610	*692
0.65	0.73	774	855	937	*019	*101	*183	*264	*346	*428	*510
0.66	0.74	592	674	757	839	921	*003	*085	*168	*2$\bar{5}$0	*332
0.67	0.75	41$\bar{5}$	497	579	662	744	827	909	992	*07$\bar{5}$	*157
0.68	0.76	240	323	406	488	571	654	737	820	903	986
0.69	0.77	069	152	235	318	401	48$\bar{5}$	568	651	734	818
0.70		901	984	*068	*151	*23$\bar{5}$	*318	*402	*485	*569	*653
0.71	0.78	736	820	904	987	*071	*155	*239	*323	*407	*491
0.72	0.79	57$\bar{5}$	659	743	827	911	99$\bar{5}$	*079	*163	*248	*332
0.73	0.80	416	500	58$\bar{5}$	669	754	838	922	*007	*091	*176
0.74	0.81	261	345	430	51$\bar{5}$	599	684	769	854	938	*023
0.75	0.82	108	193	278	363	448	533	618	703	788	873
0.76		959	*044	*129	*214	*300	*385	*470	*556	*641	*727
0.77	0.83	812	898	983	*069	*154	*240	*325	*411	*497	*583
0.78	0.84	668	754	840	926	*012	*097	*183	*269	*355	*441
0.79	0.85	527	613	700	786	872	958	*044	*130	*217	*303
0.80	0.86	389	476	562	648	73$\bar{5}$	821	908	994	*081	*167
0.81	0.87	254	340	427	514	600	687	774	861	947	*034
0.82	0.88	121	208	29$\bar{5}$	382	469	556	643	730	817	904
0.83		991	*078	*165	*252	*339	*427	*514	*601	*689	*776
0.84	0.89	863	951	*038	*125	*213	*300	*388	*475	*563	*651
0.85	0.90	738	826	914	*001	*089	*177	*264	*352	*440	*528
0.86	0.91	616	704	791	879	967	*055	*143	*231	*319	*408
0.87	0.92	496	584	672	760	848	936	*02$\bar{5}$	*113	*201	*290
0.88	0.93	378	466	55$\bar{5}$	643	732	820	908	997	*086	*174
0.89	0.94	263	351	440	529	617	706	79$\bar{5}$	883	972	*061
0.90	0.95	1$\bar{5}$0	239	327	416	505	594	683	772	861	9$\bar{5}$0
0.91	0.96	039	128	217	306	395	48$\bar{5}$	574	663	752	841
0.92		931	*020	*109	*198	*288	*377	*467	*556	*645	*73$\bar{5}$
0.93	0.97	824	914	*003	*093	*182	*272	*362	*451	*541	*631
0.94	0.98	720	810	900	989	*079	*169	*259	*349	*439	*528
0.95	0.99	618	708	798	888	978	*068	*158	*248	*338	*428
0.96	1.00	519	609	699	789	879	969	*060	*1$\bar{5}$0	*240	*330
0.97	1.01	421	511	601	692	782	873	963	*053	*144	*234
0.98	1.02	32$\bar{5}$	415	506	597	687	778	868	959	*0$\bar{5}$0	*140
0.99	1.03	231	322	413	503	594	68$\bar{5}$	776	867	957	*048
1.00	1.04	139	230	321	412	503	594	685	776	867	958
A.	B.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	74	75	76
1	7.4	7.5	7.6
2	14.8	15.0	15.2
3	22.2	22.5	22.8
4	29.6	30.0	30.4
5	37.0	37.5	38.0
6	44.4	45.0	45.6
7	51.8	52.5	53.2
8	59.2	60.0	60.8
9	66.6	67.5	68.4

	77	78	79
1	7.7	7.8	7.9
2	15.4	15.6	15.8
3	23.1	23.4	23.7
4	30.8	31.2	31.6
5	38.5	39.0	39.5
6	46.2	46.8	47.4
7	53.9	54.6	55.3
8	61.6	62.4	63.2
9	69.3	70.2	71.1

	80	81	82
1	8.0	8.1	8.2
2	16.0	16.2	16.4
3	24.0	24.3	24.6
4	32.0	32.4	32.8
5	40.0	40.5	41.0
6	48.0	48.6	49.2
7	56.0	56.7	57.4
8	64.0	64.8	65.6
9	72.0	72.9	73.8

	83	84	85
1	8.3	8.4	8.5
2	16.6	16.8	17.0
3	24.9	25.2	25.5
4	33.2	33.6	34.0
5	41.5	42.0	42.5
6	49.8	50.4	51.0
7	58.1	58.8	59.5
8	66.4	67.2	68.0
9	74.7	75.6	76.5

	86	87	88
1	8.6	8.7	8.8
2	17.2	17.4	17.6
3	25.8	26.1	26.4
4	34.4	34.8	35.2
5	43.0	43.5	44.0
6	51.6	52.2	52.8
7	60.2	60.9	61.6
8	68.8	69.6	70.4
9	77.4	78.3	79.2

	89	90	91
1	8 9	9.0	9.1
2	17.8	18.0	18.2
3	26.7	27.0	27.3
4	35.6	36.0	36.4
5	44.5	45.0	45.5
6	53.4	54.0	54.6
7	62.3	63.0	63.7
8	71.2	72.0	72.8
9	80.1	81.0	81.9

Prop. Pts.

ADD. $\begin{cases} \log a - \log b = A. \\ \log (a + b) = \log b + B. \end{cases}$ SUB. $\begin{cases} \log a - \log b = B. \\ \log (a - b) = \log b + A. \end{cases}$

A.	B.	0	1	2	3	4	5	6	7	8	9
1.00	1.04	139	230	321	412	503	594	685	776	867	958
1.01	1.05	049	140	232	323	414	505	596	687	779	870
1.02		961	*053	*144	*235	*326	*418	*509	*601	*692	*783
1.03	1.06	87$\bar{5}$	966	*058	*149	*241	*332	*424	*516	*607	*699
1.04	1.07	790	882	974	*065	*157	*249	*341	*432	*524	*616
1.05	1.08	708	800	891	983	*07$\underline{5}$	*167	*259	*351	*443	*53$\bar{5}$
1.06	1.09	627	719	811	903	99$\bar{5}$	*087	*179	*271	*363	*455
1.07	1.10	548	640	732	824	916	*009	*101	*193	*285	*378
1.08	1.11	470	562	65$\bar{5}$	747	839	932	*024	*117	*209	*301
1.09	1.12	394	486	579	671	764	857	949	*042	*134	*227
1.10	1.13	320	412	50$\bar{5}$	598	690	783	876	968	*061	*154
1.11	1.14	247	340	432	525	618	711	804	897	990	*083
1.12	1.15	175	268	361	454	547	640	733	826	920	*013
1 13	1.16	106	199	292	385	478	571	66$\bar{5}$	758	851	944
1.14	1.17	037	131	224	317	411	504	597	691	784	877
1.15		971	*064	*157	*251	*344	*438	*531	*62$\bar{5}$	*718	*812
1.16	1.18	905	999	*092	*186	*279	*373	*467	*560	*654	*748
1.17	1.19	841	93$\bar{5}$	*029	*122	*216	*310	*403	*497	*591	*68$\bar{5}$
1.18	1.20	779	872	966	*060	*154	*248	*342	*435	*529	*623
1.19	1.21	717	811	905	999	*093	*187	*281	*375	*469	*563
1.20	1.22	657	751	845	939	*034	*128	*222	*316	*410	*504
1.21	1.23	599	693	787	881	975	*070	*164	*258	*352	*447
1.22	1.24	541	635	730	824	918	*013	*107	*202	*296	*390
1.23	1.25	48$\bar{5}$	579	674	768	863	957	*052	*146	*241	*335
1.24	1.26	430	524	619	714	808	903	997	*092	*187	*281
1.25	1.27	376	471	565	660	75$\bar{5}$	8$\bar{5}$0	944	*039	*134	*229
1.26	1.28	323	418	513	608	703	797	892	987	*082	*177
1.27	1.29	272	367	462	557	652	746	841	936	*031	*126
1.28	1.30	221	316	411	507	602	697	792	887	982	*077
1.29	1.31	172	267	362	458	553	648	743	838	933	*029
1.30	1.32	124	219	314	410	50$\bar{5}$	600	695	791	886	981
1.31	1.33	077	172	267	363	458	553	649	744	840	935
1.32	1.34	030	126	221	317	412	508	603	699	794	890
1.33		985	*081	*176	*272	*367	*463	*559	*654	*7$\bar{5}$0	*845
1.34	1.35	941	*037	*132	*228	*324	*419	*515	*611	*706	*802
1.35	1.36	898	994	*089	*185	*281	*377	*472	*568	*664	*760
1.36	1.37	856	951	*047	*143	*239	*33$\bar{5}$	*431	*527	*622	*718
1.37	1.38	814	910	*006	*102	*198	*294	*390	*486	*582	*678
1.38	1.39	774	870	966	*062	*158	*254	*3$\bar{5}$0	*446	*542	*638
1.39	1.40	734	830	926	*022	*119	*21$\bar{5}$	*311	*407	*503	*599
1.40	1 41	695	792	888	984	*080	*176	*273	*369	*465	*561
1.41	1.42	658	754	850	946	*043	*139	*235	*332	*428	*524
1.42	1.43	621	717	813	910	*006	*102	*199	*295	*391	*488
1.43	1.44	584	681	777	874	970	*066	*163	*259	*356	*452
1.44	1.45	549	645	742	838	93$\bar{5}$	*031	*128	*22$\bar{5}$	*321	*418
1.45	1.46	514	611	707	804	901	997	*094	*190	*287	*384
1.46	1.47	480	577	674	770	867	964	*060	*157	*254	*350
1.47	1.48	447	544	641	737	834	931	*028	*124	*221	*318
1.48	1.49	41$\bar{5}$	512	608	705	802	899	996	*093	*189	*286
1.49	1.50	383	480	577	674	771	868	964	*061	*158	*255
1.50	1.51	352	449	546	643	740	837	934	*031	*128	*22$\bar{5}$
A.	**B.**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Pts.

	91	92
1	9.1	9.2
2	18.2	18.4
3	27.3	27.6
4	36.4	36.8
5	45.5	46.0
6	54.6	55.2
7	63.7	64.4
8	72.8	73.6
9	81.9	82.8

	93
1	9.3
2	18.6
3	27.9
4	37.2
5	46.5
6	55.8
7	65.1
8	74.4
9	83.7

	94
1	9.4
2	18.8
3	28.2
4	37.6
5	47.0
6	56.4
7	65.8
8	75.2
9	84.6

	95	96
1	9.5	9.6
2	19.0	19.2
3	28.5	28.8
4	38.0	38.4
5	47.5	48.0
6	57.0	57.6
7	66.5	67.2
8	76.0	76.8
9	85.5	86.4

	97
1	9.7
2	19.4
3	29.1
4	38.8
5	48.5
6	58.2
7	67.9
8	77.6
9	87.3

Prop. Pts.

ADD. $\begin{cases} \log a - \log b = A. \\ \log(a+b) = \log b + B. \end{cases}$ SUB. $\begin{cases} \log a - \log b = B. \\ \log(a-b) = \log b + A. \end{cases}$

A.	B.	0	1	2	3	4	5	6	7	8	9
1.50	1.51	352	449	546	643	740	837	934	*031	*128	*225̄
1.51	1.52	322	419	516	613	710	807	904	*001	*098	*195
1.52	1.53	292	389	486	583	680	778	875̄	972	*069	*166
1.53	1.54	263	360	457	555̄	652	749	846	943	*040	*138
1.54	1.55	235̄	332	429	526	624	721	818	915	*013	*110
1.55	1.56	207	304	402	499	596	693	791	888	985	*083
1.56	1.57	180	277	375̄	472	569	667	764	861	959	*056
1.57	1.58	153	251	348	446	543	640	738	835	933	*030
1.58	1.59	128	225̄	322	420	517	615̄	712	810	907	*005̄
1.59	1.60	102	200	297	395̄	492	590	687	785̄	882	980
1.60	1.61	077	175̄	273	370	468	565	663	760	858	956
1.61	1.62	053	151	248	346	444	541	639	737	834	932
1.62	1.63	030	127	225̄	322	420	518	616	713	811	909
1 63	1.64	006	104	202	299	397	495̄	593	690	788	886
1.64		984	*081	*179	*277	*375̄	*473	*570	*668	*766	*864
1.65	1.65	962	*059	*157	*255̄	*353	*451	*548	*646	*744	*842
1.66	1.66	940	*038	*136	*233	*331	*429	*527	*625	*723	*821
1.67	1.67	919	*017	*115̄	*212	*310	*408	*506	*604	*702	*800
1.68	1.68	898	996	*094	*192	*290	*388	*486	*584	*682	*780
1.69	1.69	878	976	*074	*172	*270	*368	*466	*564	*662	*760
1.70	1.70	858	956	*054	*152	*250	*348	*446	*544	*642	*741
1.71	1.71	839	937	*035̄	*133	*231	*329	*427	*525	*623	*722
1.72	1.72	820	918	*016	*114	*212	*310	*409	*507	*605̄	*703
1.73	1.73	801	899	998	*096	*194	*292	*390	*489	*587	*685̄
1.74	1.74	783	881	980	*078	*176	*274	*373	*471	*569	*667
1.75	1.75	766	864	962	*060	*159	*257	*355	*453	*552	*65̄0
1.76	1.76	748	847	945̄	*043	*141	*240	*338	*436	*535̄	*633
1.77	1.77	731	830	928	*026	*125̄	*223	*321	*420	*518	*616
1.78	1.78	715̄	813	912	*010	*108	*207	*305	*403	*502	*600
1.79	1.79	699	797	896	994	*092	*191	*289	*388	*486	*584
1.80	1.80	683	781	880	978	*077	*175	*274	*372	*471	*569
1.81	1.81	667	766	864	963	*061	*160	*258	*357	*455	*554
1.82	1.82	652	751	849	948	*046	*145̄	*244	*342	*441	*539
1.83	1.83	638	736	835̄	933	*032	*130	*229	*328	*426	*525̄
1.84	1.84	623	722	820	919	*018	*116	*215̄	*313	*412	*511
1.85	1.85	609	708	806	905	*004	*102	*201	*299	*398	*497
1.86	1.86	595	694	793	891	990	*089	*187	*286	*385̄	*483
1.87	1.87	582	681	779	878	977	*075	*174	*273	*371	*470
1.88	1.88	569	667	766	865̄	964	*062	*161	*260	*358	*457
1.89	1.89	556	655̄	753	852	951	*05̄0	*148	*247	*346	*445̄
1.90	1 90	543	642	741	840	938	*037	*136	*235̄	*333	*432
1.91	1.91	531	630	729	827	926	*025̄	*124	*223	*321	*420
1.92	1.92	519	618	717	815	914	*013	*112	*211	*310	*408
1.93	1.93	507	606	705̄	804	903	*002	*100	*199	*298	*397
1.94	1.94	496	595̄	694	792	891	990	*089	*188	*287	*386
1.95	1.95	485̄	583	682	781	880	979	*078	*177	*276	*375̄
1.96	1.96	474	573	671	770	869	968	*067	*166	*265	*364
1.97	1.97	463	562	661	760	859	958	*057	*156	*254	*353
1.98	1.98	452	551	650	749	848	947	*046	*145	*244	*343
1.99	1.99	442	541	640	739	838	937	*036	*135	*234	*333
2.00	2.00	432	531	630	729	828	927	*026	*125	*224	*323
A.	B.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	97
1	9.7
2	19.4
3	29.1
4	38.8
5	48.5
6	58.2
7	67.9
8	77.6
9	87.3

	98
1	9.8
2	19.6
3	29.4
4	39.2
5	49.0
6	58.8
7	68.6
8	78.4
9	88.2

	99
1	9.9
2	19.8
3	29.7
4	39.6
5	49.5
6	59.4
7	69.3
8	79.2
9	89.1

Prop. Pts.

$\log a - \log b = A.$ $\log a - \log b = B.$

$\log (a + b) = \log a + (B - A).$ $\log (a - b) = \log a - (B - A).$

A.	B.	B—A.	A.	B.	B—A.	A.	B.	B—A.
1.9823	1.9868	.00450	2.0337	2.0377	.00400	2.0920	2.0955	.00350
.9833	.9878	449	.0348	.0388	399	.0932	.0967	349
.9842	.9887	448	.0359	.0399	398	.0945	.0980	348
.9852	.9897	447	.0370	.0410	397	.0957	.0992	347
.9862	.9907	446	.0381	.0421	396	.0970	.1005	346
1.9872	1.9917	.00445	2.0392	2.0432	.00395	2.0982	2.1017	.00345
.9882	.9926	444	.0403	.0443	394	.0995	.1029	344
.9891	.9935	443	.0414	.0454	393	.1008	.1042	343
.9901	.9945	442	.0425	.0465	392	.1020	.1054	342
.9911	.9955	441	.0437	.0476	391	.1033	.1067	341
1.9921	1.9965	.00440	2.0448	2.0487	.00390	2.1046	2.1080	.00340
.9931	.9975	439	.0459	.0498	389	.1059	.1093	339
.9941	.9985	438	.0470	.0509	388	.1072	.1106	338
.9951	.9995	437	.0481	.0520	387	.1085	.1119	337
.9961	2.0005	436	.0493	.0532	386	.1098	.1132	336
1.9971	2.0015	.00435	2.0504	2.0543	.00385	2.1111	2.1144	.00335
.9981	.0024	434	.0515	.0553	384	.1124	.1157	334
.9991	.0034	433	.0527	.0565	383	.1137	.1170	333
2.0001	.0044	432	.0538	.0576	382	.1150	.1183	332
.0011	.0054	431	.0550	.0588	381	.1163	.1196	331
2.0021	2.0065	.00430	2.0561	2.0600	.00380	2.1176	2.1209	00330
.0032	.0075	429	.0573	.0611	379	.1190	.1223	329
.0042	.0085	428	.0584	.0622	378	.1203	.1236	328
.0052	.0095	427	.0596	.0634	377	.1216	.1249	327
.0062	.0105	426	.0607	.0645	376	.1229	.1262	326
2.0073	2.0115	.00425	2.0619	2.0656	.00375	2.1243	2.1275	.00325
.0083	.0125	424	.0630	.0667	374	.1256	.1288	324
.0093	.0135	423	.0642	.0679	373	.1270	.1302	323
.0104	.0146	422	.0654	.0691	372	.1283	.1315	322
.0114	.0156	421	.0666	.0703	371	.1297	.1329	321
2.0124	2.0166	.00420	2.0677	2.0714	.00370	2.1310	2.1342	.00320
.0135	.0177	419	.0689	.0726	369	.1324	.1356	319
.0145	.0187	418	.0701	.0738	368	.1338	.1370	318
.0156	.0198	417	.0713	.0750	367	.1351	.1383	317
.0166	.0208	416	.0725	.0762	366	.1365	.1397	316
2.0177	2.0218	.00415	2.0737	2.0773	.00365	2.1379	2.1410	.00315
.0187	.0228	414	.0749	.0785	364	.1393	.1424	314
.0198	.0239	413	.0761	.0797	363	.1407	.1438	313
.0208	.0249	412	.0773	.0809	362	.1421	.1452	312
.0219	.0260	411	.0785	.0821	361	.1435	.1466	311
2.0229	2.0270	.00410	2.0797	2.0833	.00360	2.1449	2.1480	.00310
.0240	.0281	409	.0809	.0845	359	.1463	.1494	309
.0251	.0292	408	.0821	.0857	358	.1477	.1508	308
.0261	.0302	407	.0833	.0869	357	.1491	.1522	307
.0272	.0313	406	.0845	.0881	356	.1505	.1536	306
2.0283	2.0324	.00405	2.0858	2.0893	.00355	2.1520	2.1550	.00305
.0294	.0334	404	.0870	.0905	354	.1534	.1564	304
.0305	.0345	403	.0882	.0917	353	.1548	.1578	303
.0315	.0355	402	.0895	.0930	352	.1563	.1593	302
.0326	.0366	401	.0907	.0942	351	.1577	.1607	301
2.0337	2.0337	.00400	2.0920	2.0955	.00350	2.1592	2 1622	.00300
A.	B.	B—A.	A.	B.	B—A.	A.	B.	B—A.

$\log a - \log b = A.$ $\log (a + b) = \log a + (B - A).$

$\log a - \log b = B.$ $\log (a - b) = \log a - (B - A).$

A.	B.	B−A.	A.	B.	B−A.	A.	B.	B−A.
2.1592	2.1622	.00300	2.2386	2.2411	.00250	2.3358	2.3378	.00200
.1606	.1636	299	.2403	.2428	249	.3379	.3399	199
.1621	.1651	298	.2421	.2446	248	.3401	.3421	198
.1635	.1665	297	.2439	.2464	247	.3423	.3443	197
.1650	.1680	296	.2456	.2481	246	.3446	.3466	196
2.1665	2.1694	.00295	2.2474	2.2498	.00245	2.3468	2.3487	.00195
.1680	.1709	294	.2492	.2516	244	.3490	.3509	194
.1694	.1723	293	.2510	.2534	243	.3513	.3532	193
.1710	.1739	292	.2528	.2552	242	.3535	.3554	192
.1724	.1753	291	.2546	.2570	241	.3558	.3577	191
2.1739	2.1768	.00290	2.2564	2.2588	.00240	2.3581	2.3600	.00190
.1754	.1783	289	.2582	.2606	239	.3604	.3623	189
.1770	.1799	288	.2600	.2624	238	.3627	.3646	188
.1785	.1814	287	.2618	.2642	237	.3650	.3669	187
.1800	.1829	286	.2637	.2661	236	.3673	.3692	186
2.1815	2.1844	.00285	2.2656	2.2679	.00235	2.3697	2.3715	.00185
.1830	.1858	284	.2674	.2697	234	.3720	.3738	184
.1846	.1874	283	.2693	.2716	233	.3744	.3762	183
.1861	.1889	282	.2711	.2734	232	.3768	.3786	182
.1877	.1905	281	.2730	.2753	231	.3792	.3810	181
2.1892	2.1920	.00280	2.2749	2.2772	.00230	2.3816	2.3834	.00180
.1908	.1936	279	.2768	.2791	229	.3840	.3858	179
.1923	.1951	278	.2787	.2810	228	.3865	.3883	178
.1939	.1967	277	.2806	.2829	227	.3889	.3907	177
.1955	.1983	276	.2825	.2848	226	.3914	.3932	176
2.1971	2.1998	.00275	2.2845	2.2867	.00225	2.3939	2.3956	.00175
.1987	.2014	274	.2864	.2886	224	.3964	.3981	174
.2002	.2029	273	.2884	.2906	223	.3989	.4006	173
.2019	.2046	272	.2903	.2925	222	.4014	.4031	172
.2035	.2062	271	.2923	.2945	221	.4039	.4056	171
2.2051	2.2078	.00270	2.2943	2.2965	.00220	2.4065	2.4082	.00170
.2067	.2094	269	.2962	.2984	219	.4090	.4107	169
.2083	.2110	268	.2982	.3004	218	.4116	.4133	168
.2099	.2126	267	.3002	.3024	217	.4142	.4159	167
.2116	.2143	266	.3022	.3044	216	.4168	.4185	166
2.2132	2.2159	.00265	2.3043	2.3064	.00215	2.4195	2.4211	.00165
.2149	.2175	264	.3063	.3084	214	.4221	.4237	164
.2165	.2191	263	.3083	.3104	213	.4248	.4264	163
.2182	.2208	262	.3104	.3125	212	.4275	.4291	162
.2198	.2224	261	.3124	.3145	211	.4302	.4318	161
2.2215	2.2241	.00260	2.3145	2.3166	.00210	2.4329	2.4345	.00160
.2232	.2258	259	.3166	.3187	209	.4356	.4372	159
.2249	.2275	258	.3187	.3208	208	.4383	.4399	158
.2266	.2292	257	.3208	.3229	207	.4411	.4427	157
.2283	.2309	256	.3229	.3250	206	.4439	.4455	156
2.2300	2.2325	.00255	2.3250	2.3271	.00205	2.4467	2.4482	.00155
.2317	.2342	254	.3271	.3291	204	.4495	.4510	154
.2334	.2359	253	.3293	.3313	203	.4523	.4538	153
.2351	.2376	252	.3314	.3334	202	.4552	.4567	152
.2369	.2394	251	.3336	.3356	201	.4581	.4596	151
2.2386	2.2411	.00250	2.3358	2.3378	.00200	2.4609	2 4624	.00150
A.	B.	B−A.	A.	B.	B−A.	A.	B.	B−A.

$\log a - \log b = A.$ $\log a - \log b = B.$

$\log (a + b) = \log a + (B - A).$ $\log (a - b) = \log a - (B - A).$

A.	B.	B—A.	A.	B.	B—A.	A.	B.	B—A.
2.4609	2.4624	.00150	2.6373	2.6383	.00100	2.9385	2.9390	.00050
.4638	.4653	149	.6416	.6426	.00099	.9474	.9479	49
.4668	.4683	148	.6461	.6471	98	.9563	.9568	48
.4697	.4712	147	.6505	.6515	97	.9655	.9660	47
.4727	.4742	146	.6550	.6560	96	.9748	.9753	46
2.4757	2.4772	.00145	2.6596	2.6606	.00095	2.9844	2.9848	.00045
.4787	.4801	144	.6642	.6651	94	2.9941	2.9945	44
.4817	.4831	143	.6688	.6697	93	3.0041	3.0045	43
.4848	.4862	142	.6735	.6744	92	.0143	.0147	42
.4878	.4892	141	.6783	.6792	91	.0248	.0252	41
2.4910	2.4924	.00140	2.6831	2.6840	.00090	3.0356	3.0360	.00040
.4941	.4955	139	.6880	.6889	89	.0466	.0470	39
.4972	.4986	138	.6928	.6937	88	.0578	.0582	38
.5004	.5018	137	.6978	.6987	87	.0694	.0698	37
.5036	.5050	136	.7028	.7037	86	.0813	.0817	36
2.5068	2.5081	.00135	2.7079	2.7088	.00085	3.0935	3.0939	.00035
.5100	.5113	134	.7131	.7139	84	.1061	.1064	34
.5133	.5146	133	.7183	.7191	83	.1191	.1194	33
.5165	.5178	132	.7236	.7244	82	.1324	.1327	32
.5199	.5212	131	.7289	.7297	81	.1463	.1466	31
2.5232	2.5245	.00130	2.7343	2.7351	.00080	3.1606	3.1609	.00030
.5266	.5279	129	.7398	.7406	79	.1753	.1756	29
.5299	.5312	128	.7453	.7461	78	.1905	.1908	28
.5333	.5346	127	.7509	.7517	77	.2063	.2066	27
.5368	.5381	126	.7566	.7574	76	.2226	.2229	26
2.5402	2.5415	.00125	2.7623	2.7631	.00075	3.2396	3.2399	.00025
.5437	.5449	124	.7682	.7689	74	.2575	.2577	24
.5472	.5484	123	.7741	.7748	73	.2760	.2762	23
.5508	.5520	122	.7801	.7808	72	.2952	.2954	22
.5544	.5556	121	.7862	.7869	71	.3154	.3156	21
2.5580	2.5592	.00120	2.7923	2.7930	.00070	3.3366	3.3368	.00020
.5616	.5628	119	.7985	.7992	69	.3590	.3592	19
.5653	.5665	118	.8050	.8057	68	.3825	.3827	18
.5690	.5702	117	.8114	.8121	67	.4072	.4074	17
.5727	.5739	116	.8180	.8187	66	.4335	.4337	16
2.5765	2.5776	.00115	2.8245	2.8252	.00065	3.4617	3.4619	.00015
.5803	.5814	114	.8313	.8319	64	.4917	.4918	14
.5841	.5852	113	.8381	.8387	63	.5237	.5238	13
.5880	.5891	112	.8451	.8457	62	.5587	.5588	12
.5919	.5930	111	.8521	.8527	61	.5964	.5965	11
2.5958	2.5969	.00110	2.8593	2.8599	.00060	3.6377	3.6378	.00010
.5998	.6009	109	.8666	.8672	59	.6835	.6836	09
.6038	.6049	108	.8741	.8747	58	.7345	.7346	08
.6079	.6090	107	.8816	.8822	57	.7925	.7926	07
.6120	.6131	106	.8893	.8899	56	.8595	.8596	06
2.6161	2.6172	.00105	2.8971	2.8977	.00055	3.9390	3.9391	.00005
.6202	.6212	104	.9051	.9056	54	4.0355	4.0355	04
.6244	.6254	103	.9132	.9137	53	4.1600	4.1600	03
.6287	.6297	102	.9215	.9220	52	4.3375	4.3375	02
.6329	.6339	101	.9300	.9305	51	4.6367	4.6367	01
2.6373	2.6383	.00100	2.9385	2.9390	.00050	∞	∞	.00000
A.	B.	B—A.	A.	B.	B—A.	A.	B.	B—A.

TABLE VII.

SQUARES OF NUMBERS.

No.	Square.	No.	Square.	No.	Square.	No.	Square.	No.	Square.
0	0	20	400	40	1600	60	3600	80	6400
1	1	21	441	41	1681	61	3721	81	6561
2	4	22	484	42	1764	62	3844	82	6724
3	9	23	529	43	1849	63	3969	83	6889
4	16	24	576	44	1936	64	4096	84	7056
5	25	25	625	45	2025	65	4225	85	7225
6	36	26	676	46	2116	66	4356	86	7396
7	49	27	729	47	2209	67	4489	87	7569
8	64	28	784	48	2304	68	4624	88	7744
9	81	29	841	49	2401	69	4761	89	7921
10	100	30	900	50	2500	70	4900	90	8100
11	121	31	961	51	2601	71	5041	91	8281
12	144	32	1024	52	2704	72	5184	92	8464
13	169	33	1089	53	2809	73	5329	93	8649
14	196	34	1156	54	2916	74	5476	94	8836
15	225	35	1225	55	3025	75	5625	95	9025
16	256	36	1296	56	3136	76	5776	96	9216
17	289	37	1369	57	3249	77	5929	97	9409
18	324	38	1444	58	3364	78	6084	98	9604
19	361	39	1521	59	3481	79	6241	99	9801
20	400	40	1600	60	3600	80	6400	100	10000

	1◆◆	2◆◆	3◆◆	4◆◆	5◆◆	6◆◆	7◆◆	8◆◆	9◆◆		Diff.
00	100	400	900	1600	2500	3600	4900	6400	8100	00	1
01	102	404	906	1608	2510	3612	4914	6416	8118	01	3
02	104	408	912	1616	2520	3624	4928	6432	8136	04	5
03	106	412	918	1624	2530	3636	4942	6448	8154	09	7
04	108	416	924	1632	2540	3648	4956	6464	8172	16	9
05	110	420	930	1640	2550	3660	4970	6480	8190	25	11
06	112	424	936	1648	2560	3672	4984	6496	8208	36	13
07	114	428	942	1656	2570	3684	4998	6512	8226	49	15
08	116	432	948	1664	2580	3696	5012	6528	8244	64	17
09	118	436	954	1672	2590	3708	5026	6544	8262	81	19*
10	121	441	961	1681	2601	3721	5041	6561	8281	00	21
11	123	445	967	1689	2611	3733	5055	6577	8299	21	23
12	125	449	973	1697	2621	3745	5069	6593	8317	44	25
13	127	453	979	1705	2631	3757	5083	6609	8335	69	27
14	129	457	985	1713	2641	3769	5097	6625	8353	96	29*
15	132	462	992	1722	2652	3782	5112	6642	8372	25	31
16	134	466	998	1730	2662	3794	5126	6658	8390	56	33
17	136	470	1004	1738	2672	3806	5140	6674	8408	89	35*
18	139	475	1011	1747	2683	3819	5155	6691	8427	24	37
19	141	479	1017	1755	2693	3831	5169	6707	8445	61	39*
20	144	484	1024	1764	2704	3844	5184	6724	8464	00	41
21	146	488	1030	1772	2714	3856	5198	6740	8482	41	43
22	148	492	1036	1780	2724	3868	5212	6756	8500	84	45*
23	151	497	1043	1789	2735	3881	5227	6773	8519	29	47
24	153	501	1049	1797	2745	3893	5241	6789	8537	76	49*
25	156	506	1056	1806	2756	3906	5256	6806	8556	25	51
26	158	510	1062	1814	2766	3918	5270	6822	8574	76	53*
27	161	515	1069	1823	2777	3931	5285	6839	8593	29	55
28	163	519	1075	1831	2787	3943	5299	6855	8611	84	57*
29	166	524	1082	1840	2798	3956	5314	6872	8630	41	59*
30	169	529	1089	1849	2809	3969	5329	6889	8649	00	61
31	171	533	1095	1857	2819	3981	5343	6905	8667	61	63*
32	174	538	1102	1866	2830	3994	5358	6922	8686	24	65
33	176	542	1108	1874	2840	4006	5372	6938	8704	89	67*
34	179	547	1115	1883	2851	4019	5387	6955	8723	56	69*
35	182	552	1122	1892	2862	4032	5402	6972	8742	25	71
36	184	556	1128	1900	2872	4044	5416	6988	8760	96	73*
37	187	561	1135	1909	2883	4057	5431	7005	8779	69	75*
38	190	566	1142	1918	2894	4070	5446	7022	8798	44	77*
39	193	571	1149	1927	2905	4083	5461	7039	8817	21	79*
40	196	576	1156	1936	2916	4096	5476	7056	8836	00	81
41	198	580	1162	1944	2926	4108	5490	7072	8854	81	83*
42	201	585	1169	1953	2937	4121	5505	7089	8873	64	85*
43	204	590	1176	1962	2948	4134	5520	7106	8892	49	87*
44	207	595	1183	1971	2959	4147	5535	7123	8911	36	89*
45	210	600	1190	1980	2970	4160	5550	7140	8930	25	91*
46	213	605	1197	1989	2981	4173	5565	7157	8949	16	93*
47	216	610	1204	1998	2992	4186	5580	7174	8968	09	95*
48	219	615	1211	2007	3003	4199	5595	7191	8987	04	97*
49	222	620	1218	2016	3014	4212	5610	7208	9006	01	99*
50	225	625	1225	2025	3025	4225	5625	7225	9025	00	

	1◆◆	2◆◆	3◆◆	4◆◆	5◆◆	6◆◆	7◆◆	8◆◆	9◆◆		Diff.
50	225	625	1225	2025	3025	4225	5625	7225	9025	00	1
51	228	630	1232	2034	3036	4238	5640	7242	9044	01	3
52	231	635	1239	2043	3047	4251	5655	7259	9063	04	5
53	234	640	1246	2052	3058	4264	5670	7276	9082	09	7
54	237	645	1253	2061	3069	4277	5685	7293	9101	16	9
55	240	650	1260	2070	3080	4290	5700	7310	9120	25	11
56	243	655	1267	2079	3091	4303	5715	7327	9139	36	13
57	246	660	1274	2088	3102	4316	5730	7344	9158	49	15
58	249	665	1281	2097	3113	4329	5745	7361	9177	64	17
59	252	670	1288	2106	3124	4342	5760	7378	9196	81	19*
60	256	676	1296	2116	3136	4356	5776	7396	9216	00	21
61	259	681	1303	2125	3147	4369	5791	7413	9235	21	23
62	262	686	1310	2134	3158	4382	5806	7430	9254	44	25
63	265	691	1317	2143	3169	4395	5821	7447	9273	69	27
64	268	696	1324	2152	3180	4408	5836	7464	9292	96	29*
65	272	702	1332	2162	3192	4422	5852	7482	9312	25	31
66	275	707	1339	2171	3203	4435	5867	7499	9331	56	33
67	278	712	1346	2180	3214	4448	5882	7516	9350	89	35*
68	282	718	1354	2190	3226	4462	5898	7534	9370	24	37
69	285	723	1361	2199	3237	4475	5913	7551	9389	61	39*
70	289	729	1369	2209	3249	4489	5929	7569	9409	00	41
71	292	734	1376	2218	3260	4502	5944	7586	9428	41	43
72	295	739	1383	2227	3271	4515	5959	7603	9447	84	45*
73	299	745	1391	2237	3283	4529	5975	7621	9467	29	47
74	302	750	1398	2246	3294	4542	5990	7638	9486	76	49*
75	306	756	1406	2256	3306	4556	6006	7656	9506	25	51
76	309	761	1413	2265	3317	4569	6021	7673	9525	76	53*
77	313	767	1421	2275	3329	4583	6037	7691	9545	29	55
78	316	772	1428	2284	3340	4596	6052	7708	9564	84	57*
79	320	778	1436	2294	3352	4610	6068	7726	9584	41	59*
80	324	784	1444	2304	3364	4624	6084	7744	9604	00	61
81	327	789	1451	2313	3375	4637	6099	7761	9623	61	63*
82	331	795	1459	2323	3387	4651	6115	7779	9643	24	65
83	334	800	1466	2332	3398	4664	6130	7796	9662	89	67*
84	338	806	1474	2342	3410	4678	6146	7814	9682	56	69*
85	342	812	1482	2352	3422	4692	6162	7832	9702	25	71
86	345	817	1489	2361	3433	4705	6177	7849	9721	96	73*
87	349	823	1497	2371	3445	4719	6193	7867	9741	69	75*
88	353	829	1505	2381	3457	4733	6209	7885	9761	44	77*
89	357	835	1513	2391	3469	4747	6225	7903	9781	21	79*
90	361	841	1521	2401	3481	4761	6241	7921	9801	00	81
91	364	846	1528	2410	3492	4774	6256	7938	9820	81	83*
92	368	852	1536	2420	3504	4788	6272	7956	9840	64	85*
93	372	858	1544	2430	3516	4802	6288	7974	9860	49	87*
94	376	864	1552	2440	3528	4816	6304	7992	9880	36	89*
95	380	870	1560	2450	3540	4830	6320	8010	9900	25	91*
96	384	876	1568	2460	3552	4844	6336	8028	9920	16	93*
97	388	882	1576	2470	3564	4858	6352	8046	9940	09	95*
98	392	888	1584	2480	3576	4872	6368	8064	9960	04	97*
99	396	894	1592	2490	3588	4886	6384	8082	9980	01	99*
100	400	900	1600	2500	3600	4900	6400	8100	10000	00	

TABLE VIII.—DECIMALS OF DAY INTO HOURS, ETC.

D.	H. M. S.	H. M. S. / 100	H.M.S. / 100^2	D.	H. M. S.	H. M. S. / 100	H.M.S. / 100^2
d.	*h. m. s.*	*m. s.*	*s.*	*d.*	*h. m. s.*	*m. s.*	*s.*
0.01	0 14 24	0 8.64	0.09	0.51	12 14 24	7 20.64	4.41
0.02	0 28 48	0 17.28	0.17	0.52	12 28 48	7 29.28	4.49
0.03	0 43 12	0 25.92	0.26	0.53	12 43 12	7 37.92	4.58
0.04	0 57 36	0 34.56	0.35	0.54	12 57 36	7 46.56	4.67
0.05	1 12 0	0 43.20	0.43	0.55	13 12 0	7 55.20	4.75
0.06	1 26 24	0 51.84	0.52	0.56	13 26 24	8 3.84	4.84
0.07	1 40 48	1 0.48	0.60	0.57	13 40 48	8 12.48	4.92
0.08	1 55 12	1 9.12	0.69	0.58	13 55 12	8 21.12	5.01
0.09	2 9 36	1 17.76	0.78	0.59	14 9 36	8 29.76	5.10
0.10	2 24 0	1 26.40	0.86	0.60	14 24 0	8 38.40	5.18
0.11	2 38 24	1 35.04	0.95	0.61	14 38 24	8 47.04	5.27
0.12	2 52 48	1 43.68	1.04	0.62	14 52 48	8 55.68	5.36
0.13	3 7 12	1 52.32	1.12	0.63	15 7 12	9 4.32	5.44
0.14	3 21 36	2 0.96	1.21	0.64	15 21 36	9 12.96	5.53
0.15	3 36 0	2 9.60	1.30	0.65	15 36 0	9 21.60	5.62
0.16	3 50 24	2 18.24	1.38	0.66	15 50 24	9 30.24	5.70
0.17	4 4 48	2 26.88	1.47	0.67	16 4 48	9 38.88	5.79
0.18	4 19 12	2 35.52	1.56	0.68	16 19 12	9 47.52	5.88
0.19	4 33 36	2 44.16	1.64	0.69	16 33 36	9 56.16	5.96
0.20	4 48 0	2 52.80	1.73	0.70	16 48 0	10 4.80	6.05
0.21	5 2 24	3 1.44	1.81	0.71	17 2 24	10 13.44	6.13
0.22	5 16 48	3 10.08	1.90	0.72	17 16 48	10 22.08	6.22
0.23	5 31 12	3 18.72	1.99	0.73	17 31 12	10 30.72	6.31
0.24	5 45 36	3 27.36	2.07	0.74	17 45 36	10 39.36	6.39
0.25	6 0 0	3 36.00	2.16	0.75	18 0 0	10 48.00	6.48
0.26	6 14 24	3 44.64	2.25	0.76	18 14 24	10 56.64	6.57
0.27	6 28 48	3 53.28	2.33	0.77	18 28 48	11 5.28	6.65
0.28	6 43 12	4 1.92	2.42	0.78	18 43 12	11 13.92	6.74
0.29	6 57 36	4 10.56	2.51	0.79	18 57 36	11 22.56	6.83
0.30	7 12 0	4 19.20	2.59	0.80	19 12 0	11 31.20	6.91
0.31	7 26 24	4 27.84	2.68	0.81	19 26 24	11 39.84	7.00
0.32	7 40 48	4 36.48	2.76	0.82	19 40 48	11 48.48	7.08
0.33	7 55 12	4 45.12	2.85	0.83	19 55 12	11 57.12	7.17
0.34	8 9 36	4 53.76	2.94	0.84	20 9 36	12 5.76	7.26
0.35	8 24 0	5 2.40	3.02	0.85	20 24 0	12 14.40	7.34
0.36	8 38 24	5 11.04	3.11	0.86	20 38 24	12 23.04	7.43
0.37	8 52 48	5 19.68	3.20	0.87	20 52 48	12 31.68	7.52
0.38	9 7 12	5 28.32	3.28	0.88	21 7 12	12 40.32	7.60
0.39	9 21 36	5 36.96	3.37	0.89	21 21 36	12 48.96	7.69
0.40	9 36 0	5 45.60	3.46	0.90	21 36 0	12 57.60	7.78
0.41	9 50 24	5 54.24	3.54	0.91	21 50 24	13 6.24	7.86
0.42	10 4 48	6 2.88	3.63	0.92	22 4 48	13 14.88	7.95
0.43	10 19 12	6 11.52	3.72	0.93	22 19 12	13 23.52	8.04
0.44	10 33 36	6 20.16	3.80	0.94	22 33 36	13 32.16	8.12
0.45	10 48 0	6 28.80	3.89	0.95	22 48 0	13 40.80	8.21
0.46	11 2 24	6 37.44	3.97	0.96	23 2 24	13 49.44	8.29
0.47	11 16 48	6 46.08	4.06	0.97	23 16 48	13 58.08	8.38
0.48	11 31 12	6 54.72	4.15	0.98	23 31 12	14 6.72	8.47
0.49	11 45 36	7 3.36	4.23	0.99	23 45 36	14 15.36	8.55
0.50	12 0 0	7 12.00	4.32	1.00	24 0 0	14 24.00	8.64

°	h. m.	°	h. m.	°	h. m.	°	h. m.	°	h. m.	°	h. m.	'	m. s.	"	s.
0	0 0	60	4 0	120	8 0	180	12 0	240	16 0	300	20 0	0	0 0	0	0.000
1	0 4	61	4 4	121	8 4	181	12 4	241	16 4	301	20 4	1	0 4	1	0.066
2	0 8	62	4 8	122	8 8	182	12 8	242	16 8	302	20 8	2	0 8	2	0.133
3	0 12	63	4 12	123	8 12	183	12 12	243	16 12	303	20 12	3	0 12	3	0.200
4	0 16	64	4 16	124	8 16	184	12 16	244	16 16	304	20 16	4	0 16	4	0.266
5	0 20	65	4 20	125	8 20	185	12 20	245	16 20	305	20 20	5	0 20	5	0.333
6	0 24	66	4 24	126	8 24	186	12 24	246	16 24	306	20 24	6	0 24	6	0.400
7	0 28	67	4 28	127	8 28	187	12 28	247	16 28	307	20 28	7	0 28	7	0.466
8	0 32	68	4 32	128	8 32	188	12 32	248	16 32	308	20 32	8	0 32	8	0.533
9	0 36	69	4 36	129	8 36	189	12 36	249	16 36	309	20 36	9	0 36	9	0.600
10	0 40	70	4 40	130	8 40	190	12 40	250	16 40	310	20 40	10	0 40	10	0.666
11	0 44	71	4 44	131	8 44	191	12 44	251	16 44	311	20 44	11	0 44	11	0.733
12	0 48	72	4 48	132	8 48	192	12 48	252	16 48	312	20 48	12	0 48	12	0.800
13	0 52	73	4 52	133	8 52	193	12 52	253	16 52	313	20 52	13	0 52	13	0.866
14	0 56	74	4 56	134	8 56	194	12 56	254	16 56	314	20 56	14	0 56	14	0.933
15	1 0	75	5 0	135	9 0	195	13 0	255	17 0	315	21 0	15	1 0	15	1.000
16	1 4	76	5 4	136	9 4	196	13 4	256	17 4	316	21 4	16	1 4	16	1.066
17	1 8	77	5 8	137	9 8	197	13 8	257	17 8	317	21 8	17	1 8	17	1.133
18	1 12	78	5 12	138	9 12	198	13 12	258	17 12	318	21 12	18	1 12	18	1.200
19	1 16	79	5 16	139	9 16	199	13 16	259	17 16	319	21 16	19	1 16	19	1.266
20	1 20	80	5 20	140	9 20	200	13 20	260	17 20	320	21 20	20	1 20	20	1.333
21	1 24	81	5 24	141	9 24	201	13 24	261	17 24	321	21 24	21	1 24	21	1.400
22	1 28	82	5 28	142	9 28	202	13 28	262	17 28	322	21 28	22	1 28	22	1.466
23	1 32	83	5 32	143	9 32	203	13 32	263	17 32	323	21 32	23	1 32	23	1.533
24	1 36	84	5 36	144	9 36	204	13 36	264	17 36	324	21 36	24	1 36	24	1.600
25	1 40	85	5 40	145	9 40	205	13 40	265	17 40	325	21 40	25	1 40	25	1.666
26	1 44	86	5 44	146	9 44	206	13 44	266	17 44	326	21 44	26	1 44	26	1.733
27	1 48	87	5 48	147	9 48	207	13 48	267	17 48	327	21 48	27	1 48	27	1.800
28	1 52	88	5 52	148	9 52	208	13 52	268	17 52	328	21 52	28	1 52	28	1.866
29	1 56	89	5 56	149	9 56	209	13 56	269	17 56	329	21 56	29	1 56	29	1.933
30	2 0	90	6 0	150	10 0	210	14 0	270	18 0	330	22 0	30	2 0	30	2.000
31	2 4	91	6 4	151	10 4	211	14 4	271	18 4	331	22 4	31	2 4	31	2.066
32	2 8	92	6 8	152	10 8	212	14 8	272	18 8	332	22 8	32	2 8	32	2.133
33	2 12	93	6 12	153	10 12	213	14 12	273	18 12	333	22 12	33	2 12	33	2.200
34	2 16	94	6 16	154	10 16	214	14 16	274	18 16	334	22 16	34	2 16	34	2.266
35	2 20	95	6 20	155	10 20	215	14 20	275	18 20	335	22 20	35	2 20	35	2.333
36	2 24	96	6 24	156	10 24	216	14 24	276	18 24	336	22 24	36	2 24	36	2.400
37	2 28	97	6 28	157	10 28	217	14 28	277	18 28	337	22 28	37	2 28	37	2.466
38	2 32	98	6 32	158	10 32	218	14 32	278	18 32	338	22 32	38	2 32	38	2.533
39	2 36	99	6 36	159	10 36	219	14 36	279	18 36	339	22 36	39	2 36	39	2.600
40	2 40	100	6 40	160	10 40	220	14 40	280	18 40	340	22 40	40	2 40	40	2.666
41	2 44	101	6 44	161	10 44	221	14 44	281	18 44	341	22 44	41	2 44	41	2.733
42	2 48	102	6 48	162	10 48	222	14 48	282	18 48	342	22 48	42	2 48	42	2.800
43	2 52	103	6 52	163	10 52	223	14 52	283	18 52	343	22 52	43	2 52	43	2.866
44	2 56	104	6 56	164	10 56	224	14 56	284	18 56	344	22 56	44	2 56	44	2.933
45	3 0	105	7 0	165	11 0	225	15 0	285	19 0	345	23 0	45	3 0	45	3.000
46	3 4	106	7 4	166	11 4	226	15 4	286	19 4	346	23 4	46	3 4	46	3.066
47	3 8	107	7 8	167	11 8	227	15 8	287	19 8	347	23 8	47	3 8	47	3.133
48	3 12	108	7 12	168	11 12	228	15 12	288	19 12	348	23 12	48	3 12	48	3.200
49	3 16	109	7 16	169	11 16	229	15 16	289	19 16	349	23 16	49	3 16	49	3.266
50	3 20	110	7 20	170	11 20	230	15 20	290	19 20	350	23 20	50	3 20	50	3.333
51	3 24	111	7 24	171	11 24	231	15 24	291	19 24	351	23 24	51	3 24	51	3.400
52	3 28	112	7 28	172	11 28	232	15 28	292	19 28	352	23 28	52	3 28	52	3.466
53	3 32	113	7 32	173	11 32	233	15 32	293	19 32	353	23 32	53	3 32	53	3.533
54	3 36	114	7 36	174	11 36	234	15 36	294	19 36	354	23 36	54	3 36	54	3.600
55	3 40	115	7 40	175	11 40	235	15 40	295	19 40	355	23 40	55	3 40	55	3.666
56	3 44	116	7 44	176	11 44	236	15 44	296	19 44	356	23 44	56	3 44	56	3.733
57	3 48	117	7 48	177	11 48	237	15 48	297	19 48	357	23 48	57	3 48	57	3.800
58	3 52	118	7 52	178	11 52	238	15 52	298	19 52	358	23 52	58	3 52	58	3.866
59	3 56	119	7 56	179	11 56	239	15 56	299	19 56	359	23 56	59	3 56	59	3.933

TABLE X*a*.—TO CONVERT MEAN INTO SIDEREAL TIME.

Mean T.		Correction. +		Mean T.		Correction. +		Mean T.		Correction. +	
h.	*m.*	*m.*	*s.*	*h.*	*m.*	*m.*	*s.*	*h.*	*m.*	*m.*	*s.*
0	0	0	0.00	8	0	1	18.85	16	0	2	37.70
	10		1.64		10		20.50		10		39.35
	20		3.29		20		22.14		20		40.99
	30		4.93		30		23.78		30		42.63
	40		6.57		40		25.42		40		44.28
	50		8.21		50		27.07		50		45.92
1	0	0	9.86	9	0	1	28.71	17	0	2	47.56
	10		11.50		10		30.35		10		49.20
	20		13.14		20		31.99		20		50.85
	30		14.78		30		33.64		30		52.49
	40		16.43		40		35.28		40		54.13
	50		18.07		50		36.92		50		55.77
2	0	0	19.71	10	0	1	38.56	18	0	2	57.42
	10		21.36		10		40.21		10		59.06
	20		23.00		20		41.85		20	3	0.70
	30		24.64		30		43.49		30		2.34
	40		26.28		40		45.14		40		3.99
	50		27.93		50		46.78		50		5.63
3	0	0	29.57	11	0	1	48.42	19	0	3	7.27
	10		31.21		10		50.06		10		8.92
	20		32.86		20		51.71		20		10.56
	30		34.50		30		53.35		30		12.20
	40		36.14		40		54.99		40		13.84
	50		37.78		50		56.64		50		15.49
4	0	0	39.43	12	0	1	58.28	20	0	3	17.13
	10		41.07		10		59.92		10		18.77
	20		42.71		20	2	1.56		20		20.42
	30		44.35		30		3.21		30		22.06
	40		46.00		40		4.85		40		23.70
	50		47.64		50		6.49		50		25.34
5	0	0	49.28	13	0	2	8.13	21	0	3	26.99
	10		50.92		10		9.78		10		28.63
	20		52.57		20		11.42		20		30.27
	30		54.21		30		13.06		30		31.91
	40		55.85		40		14.70		40		33.56
	50		57.50		50		16.35		50		35.20
6	0	0	59.14	14	0	2	17.99	22	0	3	36.84
	10	1	0.78		10		19.63		10		38.48
	20		2.42		20		21.28		20		40.13
	30		4.07		30		22.92		30		41.77
	40		5.71		40		24.56		40		43.41
	50		7.35		50		26.20		50		45.06
7	0	1	9.00	15	0	2	27.85	23	0	3	46.70
	10		10.64		10		29.49		10		48.34
	20		12.28		20		31.13		20		49.98
	30		13.92		30		32.77		30		51.63
	40		15.57		40		34.42		40		53.27
	50		17.21		50		36.06		50		54.91

Corr. for min. and sec.		
m.	*s.*	*s.*
0	10	0.03
	20	0.05
	30	0.08
	40	0.11
	50	0.14
1	0	0.16
	10	0.19
	20	0.22
	30	0.25
	40	0.27
	50	0.30
2	0	0.33
	10	0.36
	20	0.38
	30	0.41
	40	0.44
	50	0.47
3	0	0.49
	10	0.52
	20	0.55
	30	0.57
	40	0.60
	50	0.63
4	0	0.66
	10	0.68
	20	0.71
	30	0.74
	40	0.77
	50	0.79
5	0	0.82
	10	0.85
	20	0.88
	30	0.90
	40	0.93
	50	0.96
6	0	0.99
	10	1.01
	20	1.04
	30	1.07
	40	1.10
	50	1.12
7	0	1.15
	10	1.18
	20	1.21
	30	1.23
	40	1.26
	50	1.29
8	0	1.31
	10	1.34
	20	1.37
	30	1.40
	40	1.42
	50	1.45
9	0	1.48
	10	1.50
	20	1.53
	30	1.56
	40	1.59
	50	1.62

TABLE X*b*.—TO CONVERT SIDEREAL INTO MEAN TIME.

Sid. T.		Correction. —		Sid. T.		Correction. —		Sid. T.		Correction. —	
h.	*m.*	*m.*	*s.*	*h.*	*m.*	*m.*	*s.*	*h.*	*m.*	*m.*	*s.*
0	0	0	0.00	8	0	1	18.64	16	0	2	37.27
	10		1.64		10		20.28		10		38.91
	20		3.28		20		21.91		20		40.55
	30		4.92		30		23.55		30		42.19
	40		6.55		40		25.19		40		43.83
	50		8.19		50		26.83		50		45.46
1	0	0	9.83	9	0	1	28.47	17	0	2	47.10
	10		11.47		10		30.10		10		48.74
	20		13.11		20		31.74		20		50.38
	30		14.74		30		33.38		30		52.02
	40		16.38		40		35.02		40		53.66
	50		18.02		50		36.66		50		55.29
2	0	0	19.66	10	0	1	38.30	18	0	2	56.93
	10		21.30		10		39.93		10		58.57
	20		22.94		20		41.57		20	3	0.21
	30		24.57		30		43.21		30		1.85
	40		26.21		40		44.85		40		3.48
	50		27.85		50		46.49		50		5.12
3	0	0	29.49	11	0	1	48.12	19	0	3	6.76
	10		31.13		10		49.76		10		8.40
	20		32.76		20		51.40		20		10.04
	30		34.40		30		53.04		30		11.68
	40		36.04		40		54.68		40		13.32
	50		37.68		50		56.32		50		14.95
4	0	0	39.32	12	0	1	57.96	20	0	3	16.59
	10		40.96		10		59.59		10		18.23
	20		42.60		20	2	1.23		20		19.87
	30		44.23		30		2.87		30		21.51
	40		45.87		40		4.51		40		23.14
	50		47.51		50		6.15		50		24.78
5	0	0	49.15	13	0	2	7.78	21	0	3	26.42
	10		50.79		10		9.42		10		28.06
	20		52.42		20		11.06		20		29.70
	30		54.06		30		12.70		30		31.34
	40		55.70		40		14.34		40		32.97
	50		57.34		50		15.98		50		34.61
6	0	0	58.98	14	0	2	17.61	22	0	3	36.25
	10	1	0.62		10		19.25		10		37.89
	20		2.25		20		20.89		20		39.53
	30		3.89		30		22.53		30		41.16
	40		5.53		40		24.17		40		42.80
	50		7.17		50		25.80		50		44.44
7	0	1	8.81	15	0	2	27.44	23	0	3	46.08
	10		10.44		10		29.08		10		47.72
	20		12.08		20		30.72		20		49.36
	30		13.72		30		32.36		30		51.00
	40		15.36		40		34.00		40		52.63
	50		17.00		50		35.64		50		54.27

Corr. for min. and sec.		
m.	*s.*	*s.*
0	10	0.03
	20	0.05
	30	0.08
	40	0.11
	50	0.14
1	0	0.16
	10	0.19
	20	0.22
	30	0.25
	40	0.27
	50	0.30
2	0	0.33
	10	0.35
	20	0.38
	30	0.41
	40	0.44
	50	0.47
3	0	0.49
	10	0.52
	20	0.55
	30	0.57
	40	0.60
	50	0.63
4	0	0.66
	10	0.68
	20	0.71
	30	0.74
	40	0.76
	50	0.79
5	0	0.82
	10	0.85
	20	0.87
	30	0.90
	40	0.93
	50	0.96
6	0	0.98
	10	1.01
	20	1.04
	30	1.06
	40	1.09
	50	1.12
7	0	1.15
	10	1.17
	20	1.20
	30	1.23
	40	1.26
	50	1.28
8	0	1.31
	10	1.34
	20	1.37
	30	1.39
	40	1.42
	50	1.45
9	0	1.47
	10	1.50
	20	1.53
	30	1.56
	40	1.58
	50	1.61

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